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Why Governments Should Target a Budget Deficit Ratio, Not a Primary Surplus Ratio When the Interest Rate on Government Bonds Increases with the Public Debt Ratio

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Abstract

Whereas a budget-deficit-to-GDP ratio is targeted in the Maastricht Treaty, a primary surplus-to-GDP ratio is targeted in the Growth and Stability Pact as well as in the Treaty on Stability Coordination and Governance concerning the adjustment programmes for the Eurozone programme countries Greece, Ireland, Portugal, and Spain. An argument for the latter is that, by implication, the government’s intertemporal budget constraint, i.e. its solvency constraint, is satisfied, and hence public debt is long-term sustainable in the sense of Englmann 2016. In the present paper we will see that the stability of the steady state and hence medium-term sustainability of public debt becomes an issue once the safety discount factor, which is defined as the difference between the rate of return on private capital and the rate of interest on government bonds divided by the rate of return on private capital, varies with the debt-to-GDP ratio. Whereas the steady state is stable if the government varies the tax rate in order to target a specific budget deficit-to-GDP ratio, in general, the steady state becomes unstable if the government targets a specific primary surplus-to-GDP ratio.

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1. Introduction

Whereas a budget-deficit-to-GDP ratio is targeted in the Maastricht Treaty, a primary surplus-to-GDP ratio is targeted in the Growth and Stability Pact as well as in the Treaty on Stability Coordination and Governance concerning the adjustment programmes for the Eurozone programme countries Greece, Ireland, Portugal, and Spain (see Darvas 2015, especially table 2 including the cited sources). An argument for the latter is that, by implication, the government's inter-temporal budget constraint, i.e. its solvency constraint, is satisfied, which means that the programme countries can really pay back their debt. Another motivation for using a primary surplus-to-GDP ratio as the target in programme countries may be that, to a large extent, the rates of interest on public debt are lower than they would be, were they not determined politically but by market forces. Furthermore, during the programme period, the government’s interest payments to institutional creditors like the IMF or the EMSF are fixed. Hence, the ratio of government’s interest payments to GDP ‘only’ varies with GDP. Accordingly, a decrease in the primary surplus-to-GDP ratio target is equivalent to an increase in the budget-deficit-to-GDP ratio (in the following also only deficit ratio) target, and vice versa.

In Englmann 2016 the following taxonomy of public debt sustainability was introduced:

<table>
<thead>
<tr>
<th>Steady state public debt dynamics</th>
<th>Government’s liquidity constraint</th>
<th>Stability conditions</th>
<th>Government’s solvency constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short-term unsustainable</td>
<td>violated in the steady state</td>
<td>violated or satisfied in the steady state</td>
<td>violated or satisfied in the steady state</td>
</tr>
<tr>
<td>Short-term sustainable yet medium-term unsustainable</td>
<td>satisfied in the steady state</td>
<td>violated or satisfied in the steady state</td>
<td>violated in the steady state</td>
</tr>
<tr>
<td>Medium-term unsustainable</td>
<td>violated in the steady state</td>
<td>violated in the steady state</td>
<td>violated or satisfied in the steady state</td>
</tr>
<tr>
<td>Medium-term sustainable yet long-term unsustainable</td>
<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
<td>violated in the steady state</td>
</tr>
<tr>
<td>Long-term sustainable</td>
<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
</tr>
</tbody>
</table>

Table 1: Categories of sustainable and unsustainable public debt dynamics

The categories of sustainable and unsustainable public debt dynamics in Table 1 explicitly refer to steady states. Hence, the existence of at least one steady state with a positive public debt-to-GDP ratio is a precondition for public debt being sustainable. According to this categorization, an ever decreasing debt ratio without a positive steady state debt ratio is not sustainable. The government’s liquidity or instantaneous budget constraint requires that in any time period the government’s revenue from taxes and other transfers as well as from a budget deficit suffices to pay for the government’s expenditures on public consumption and investment as well as for public debt service. A steady state and hence a steady state debt ratio is (asymptotically) stable if the economy returns to the steady state after a shock. The government’s solvency or intertemporal budget constraint requires the present value of the government’s primary balance to exceed its debt.

Furthermore, Englmann 2016 showed that the steady state in a Solow growth model with public debt can be long-term sustainable if the ratio of the rate of interest on government bonds to the rate of return on private capital is exogenously fixed at a level sufficiently close to unity. In this paper we assume this ratio increases with an increase in the public debt-to-GDP ratio (in the following also simply debt ratio). Under certain assumptions concerning the functional form of this ratio this leads to an positive relationship between the rate of interest on government bonds and the debt ratio, a relationship that is well documented.
in the empirical literature (see e.g. Laubach 2009 as well as Greenlaw et al 2013 and the literature referred to therein).

As will be shown below, under these assumptions stability of the steady state and hence sustainability of public debt becomes an issue if both the liquidity and especially the solvency constraint are to be satisfied. It makes a difference whether the government sets a constant tax rate, targets a budget deficit-to-GDP ratio or a primary surplus-to-GDP ratio. Whereas the steady state can be stabilized by the government if it varies the tax rate in order to target a specific budget deficit-to-GDP ratio, in general, the steady state is unstable if the government targets a specific primary surplus-to-GDP ratio. In order to deal with the question of stability of public debt dynamics a growth model of the Solow type is chosen because its relevant steady state is asymptotically stable. Thus, the conditions under which the originally stable steady state becomes unstable can be studied. This research strategy could not be followed if the starting point were a model whose steady state is either asymptotically unstable as a growth model of the Harrod-Domar type or a saddle point as an optimal growth model of the Ramsey type. This research strategy is in line with the talk that the chief economist of the IMF Olivier Blanchard gave at the ASSA meeting in January 2015. There he argued to use small non-linear models in order to cope with financial and debt crises.

The remainder of the paper is organized as follows: In section 2 the simple Solovian growth model with public debt is presented briefly. Here private households’ propensity to invest in government bonds is assumed to be an exogenous parameter just as the safety discount factor. In section 3 we will endogenize the safety discount factor in order to examine under which conditions the steady state public debt-to-GDP ratio may become unstable. In section 4 it is shown that a policy targeting a specific budget deficit-to-GDP ratio stabilizes the steady state, whereas this is not the case if a specific primary surplus-to-GDP ratio is targeted. Section 5 concludes.

2. A Simple Solovian Growth Model with Public Debt

In an accompanying paper (Englmann 2016) a simple Solovian growth model with public debt, public capital, exogenous propensity to invest in government bonds, and exogenous safety discount factor is presented. This section summarizes the main points and key equations. For more details see Englmann 2016.

Following Aschauer 1989 public capital $K_g$ is taken into account in the aggregate production function of Cobb-Douglas type:

$$
Y = \gamma A^\alpha L^\beta K_g^\delta K_{pr}^{1-\alpha-\beta} ; \quad \gamma \geq 1
$$

(1),

where $Y$ denotes aggregate net production of goods and services produced in the economy, $\gamma$ a positive constant parameter, $A$ technical efficiency, $L$ labor input, $K_{pr}$ private capital. For the sake of simplicity, the dependence of the variables on time is omitted in general. Total factor productivity grows at the rate of technical progress $\delta$, labor supply at the natural rate $n$. Labor is supplied inelastically with respect to the real net wage. Capital and labor are fully employed. Aggregate supply of goods and services equals aggregate demand consisting of private ($C_{pr}$) and public consumption ($C_g$) as well as of net private ($I_{pr}$) and net public investment ($I_g$). Private consumption depends on available income of households ($Y - T + r_B B$)

$$
C_{pr} = c_{pr} (Y - T + r_B B)
$$

(2),

$^{1}$ Domar 1944 is a seminal paper for the discussion of the sustainability of public debt in the context of economic growth.
where \( c_{pr} \) denotes households' propensity to consume with respect to disposable income, \( T \) net transfer payments from households to government (taxes plus social contributions from private households minus government's transfer payments to private households), and \( r_B \) the rate of interest on government bonds \( B \). The non-consuming part of households' disposable income is saved and one part of savings \( S_{pr}^f \) is invested in bonds/equities issued by private firms

\[
S_{pr}^f = s_{pr}^f (Y - T + r_B B) \tag{3}
\]

and the other in government bonds

\[
S_{pr}^g = s_{pr}^g (Y - T + r_B B) \tag{4}
\]

In the following, for brevity's sake \( T \) will be called taxes, and the corresponding rate \( \tau \) tax rate.

Taxes depend on the tax rate \( \tau \) and gross household income \((Y + r_B B)\), which flows from private firms as labor and capital income and from government as interest payments on government bonds

\[
T = \tau(Y + r_B B) \tag{5}
\]

Savings that flow to private firms \( S_{pr}^f \) are used to finance private firms' net investments \( I_{pr} \) in private real capital \( K_{pr} \)

\[
I_{pr} = K_{pr} = S_{pr}^f \tag{6}
\]

Savings that flow to government \( S_{pr}^g \) are used to buy new government bonds \( B^D \) which finance the government's budget deficit \( \dot{B} \):

\[
\dot{B} = B^D = S_{pr}^g \tag{7}
\]

According to eq. (7) the government supplies any additional government bonds that private households demand according to their disposable income and risk preferences. Thus, government budget deficits are determined by the portfolio choices of private households just as firms' investments, i.e., firms' deficits. The corresponding public-deficit-to-national-income ratio follows by simple division of eq. (7) by national income:

\[
\frac{\dot{B}}{Y} = \frac{S_{pr}^g}{Y} \tag{8}
\]

The government's liquidity constraint is:

\[
T + \dot{B} = C_g + I_g + r_B B \tag{9}
\]

Excluding monetization of public debt, the government's total revenue consists of taxes plus net increase in government debt \( B \). This revenue is used to finance public expenditures on consumption \( C_g \) and investment \( I_g \) plus interest payments on outstanding government debt \( r_B B \).

The government's primary deficit \( D \) is defined as follows:

\[
D = C_g + I_g - T \tag{10}
\]

\[\text{Here and in the following we use the following abbreviations: } \dot{x} = \frac{dx}{dt} \text{ and } \ddot{x} = \frac{d^2x}{dt^2}.\]
and the corresponding ratio of the primary deficit to national income \(d\) (in the following simply primary deficit ratio) as:
\[
d = \frac{D}{Y} = \frac{C_g + I_g - T}{Y}
\]  
(11).

From eqs (7) and (10) we obtain:
\[
\dot{B} = D + \tau g B
\]  
(12).

With the public-debt-to-national-income ratio \(b\) (in the following simply debt ratio) defined as:
\[
b = \frac{B}{Y}
\]  
(13)

we get from eqs (11) - (13):
\[
d = (\dot{B} - \tau g) b
\]  
(14).

For \(b > 0\) there is a primary deficit \((d > 0)\) if the rate of change of public debt exceeds the real rate of interest on public debt, and there is a primary surplus if the real rate of interest on public debt exceeds the rate of change of public debt.

From eqs (12), (13), (7), (5), and (4) we can also obtain another expression for the primary deficit ratio, namely:
\[
d = s_{pr} g (1 - \tau) - \tau B \left(1 - s_{pr} g (1 - \tau)\right) b
\]  
(15),

and hence with eq. (8):
\[
\frac{\dot{B}}{Y} = s_{pr} g (1 - \tau) (1 + \tau g b)
\]  
(16).

Thus, the primary deficit ratio as well as the deficit ratio do not only depend on households’ propensity to invest in government bonds and the tax rate, but on the ratio of government’s interest payments to national income as well, as the latter influence households’ disposable income.

The budget deficit can be used to finance the government’s net investments in public real capital \(K_g\), completely or partially, as well as to finance a part of public consumption or interest payments on outstanding government debt. We introduce the public-investment-to-budget-deficit ratio \(\lambda\) that is supposed to be set by the parliament when it decides on the budget:
\[
\lambda = \frac{l_g}{B}
\]  
(17).

In case of \(\lambda = 1\) the so-called Golden Rule of Public Finance is followed (see e.g. Bassetto and Sargent and 2006)\(^3\). For public capital accumulation we obtain:
\[
K_g = l_g = \lambda \dot{B}; \ \lambda > 0
\]  
(18).

From eqs (9) and (18) we can deduce:
\[
T = C_g + \tau g B + (1 - 1/\lambda) l_g
\]  
(19).

\(^3\) See in this context also Buiter 2001, Blanchard and Giavazzi 2004 and Sachverständigenrat 2007 pp. 3ff.
If the Golden Rule of Public Finance is followed, taxes on national income just serve to finance public consumption and interest payments on public debt. From eqs (5) and (19) we get for the share of public consumption in national income:

\[
\frac{C_g}{Y} = \tau - (1 - \tau)B \frac{B}{Y} - \left(1 - \frac{1}{\lambda}\right)\frac{I_g}{Y}
\]  

(20),

and from eq. (5) for the share of taxes in national income:

\[
\frac{T}{Y} = \tau(1 + B) \frac{B}{Y}
\]  

(21).

From eqs (2), (4), (6), (13), (18) and (21) the share of public consumption in national income can alternatively be expressed by the following equation:

\[
\frac{C_g}{Y} = 1 - \left(1 - (1 - \lambda)s_{pr}^g\right)(1 - \tau)(1 + B)\]  

(22).  

The share of public consumption in national income is an endogenous variable in this model which is determined by the government’s liquidity constraint.

As in Solow’s growth model we assume perfect competition in the markets for goods and services, labor and private capital. The price level is assumed to be unity. As all incomes flow to private households, only private households pay income tax. Hence, the rental price of private capital \( R \) equals the marginal productivity of private capital and the wage rate equals the marginal productivity of labor. Furthermore, we assume that the use of public capital is free of charge. This leads to profits \( \Pi \) even with perfect competition:

\[
\Pi = (1 - \alpha - \beta)Y
\]  

(23).

For the respective rate of profit \( r_{\Pi} \)

\[
r_{\Pi} = \frac{\Pi}{K_{pr}}
\]  

(24)

we can compute:

\[
r_{\Pi} = (1 - \alpha - \beta)\frac{Y}{K_{pr}}
\]  

(25),

and hence, for the overall rate of return on private capital \( r_K \)

\[
r_K = R + r_{\Pi}
\]  

(26).

From eqs (25) and (26) follows:

\[
r_K = (1 - \alpha)\frac{Y}{K_{pr}}
\]  

(27).

Even assuming perfect markets, we allow for the possibility that the rate of return on private capital exceeds the one on government bonds due to risk and liquidity considerations of private households. This implies that households consider an investment in government bonds less risky and more liquid than an investment in bonds of private enterprises. Private households’ risk and liquidity considerations are taken into account by the exogenously given safety discount factor \( \sigma \) which is defined as the difference between the rate of return on private capital and the rate of interest on government bonds divided by the rate of return on private capital:

\[\sigma = \frac{r_K - \Pi}{r_K} \]  

4 It should be noted that eqs (20) and (22) are not independent from each other.
\[
\sigma = \frac{r_K - r_B}{r_K}; \quad 0 \leq \sigma < 1
\]  
(28)

which leads to:

\[
r_B = (1 - \sigma)r_K; \quad 0 \leq \sigma < 1
\]  
(29)

By assumption, the safety discount factor does not exceed unity. Finally, we define private and public capital-labor ratios in efficiency units:

\[
\kappa_{pr} = \frac{K_{pr}}{AL}
\]  
(30),

\[
\kappa_g = \frac{K_g}{AL}
\]  
(31),

as well as net domestic product per capita in efficiency units:

\[
y = \frac{Y}{AL}
\]  
(32).

The production function can be rewritten by using eqs (32), (30) and (31):

\[
y = \frac{Y}{AL} = \gamma \kappa_{pr}^{1-\alpha-\beta} = y\left(\kappa_{pr}, \kappa_g\right)
\]  
(33).

From eq. (18) we obtain for time-invariant \(\lambda\).

\[
K_g = \lambda B
\]  
(34),

and hence according to eq. (13) for the debt ratio \(b\):

\[
b = \lambda^{-1} \gamma^{-1} \kappa_{pr}^{1-\alpha+\beta} \kappa_g
\]  
(35).

For the interest rate on public debt we can derive from eqs (27) and (29)

\[
r_B = \gamma(1-\sigma)(1-\alpha)\kappa_{pr}^{\beta-1}\kappa_g^{1-\alpha-\beta}
\]  
(36),

and hence with eqs (35) and (36) for the share of government’s interest payments in national income:

\[
r_Bb = \lambda^{-1}(1-\sigma)(1-\alpha)\frac{\kappa_g}{\kappa_{pr}}
\]  
(37),

for the shadow rate of return on the public budget deficit

\[
sr_B = \lambda \gamma K_g = \frac{1-\alpha-\beta}{b}
\]  
(38),

where \(\gamma K_g\) denotes the marginal productivity of public capital. According to eq. (38), the larger the debt ratio, the lower is the shadow rate of return on the public budget deficit. Public debt is efficient if the shadow rate of return on the public budget deficit is larger or equal to the rate of interest on government bonds. With eq. (21) we derive for the ratio of households’ disposable income to national income:

\[
\frac{Y - T + r_B B}{Y} = 1 - \tau + (1-\tau)r_Bb = (1-\tau)\left(1 + \lambda^{-1}(1-\sigma)(1-\alpha)\frac{\kappa_g}{\kappa_{pr}}\right)
\]  
(39).

From eqs (39), (3), and (4) we can deduce the shares of households’ investment in private and public bonds in national income:
\[
\frac{S^f_{pr}}{Y} = s^f_{pr} (1 - \tau) \left( 1 + \lambda^{-1}(1 - \sigma)(1 - \alpha) \frac{\kappa_g}{\kappa_{pr}} \right)
\]
(40),

\[
\frac{S^g_{pr}}{Y} = s^g_{pr} (1 - \tau) \left( 1 + \lambda^{-1}(1 - \sigma)(1 - \alpha) \frac{\kappa_g}{\kappa_{pr}} \right)
\]
(41).

Finally, from eqs (27) and (33) we derive for the rate of return on private capital:

\[
\gamma = (1 - \alpha) \frac{Y}{K_{pr}} = \gamma(1 - \alpha) \frac{\beta - 1}{\kappa_{pr}} (1 - \alpha - \beta)
\]
(42).

As shown in more detail in Englmann 2016, the following system of differential equations can finally be obtained:

\[
\dot{k}_{pr} = s^f_{pr} (1 - \tau) \gamma \left( \kappa_{pr} \beta + \lambda^{-1}(1 - \sigma)(1 - \alpha) \kappa_{pr}^{-1} \kappa_g^{-1} \kappa_{pr}^{2 - \alpha - \beta} \right) - (\delta + n) \kappa_{pr}
\]
(43),

and:

\[
\dot{k}_g = \lambda s^g_{pr} (1 - \tau) \gamma \left( \kappa_{pr} \beta + \lambda^{-1}(1 - \sigma)(1 - \alpha) \kappa_{pr}^{-1} \kappa_g^{-1} \kappa_{pr}^{2 - \alpha - \beta} \right) - (\delta + n) \kappa_g
\]
(44).

The steady state \((k_{pr}^*, k_g^*)\) is derived by setting both eqs (43) and (44) equal to zero. By dividing the two resulting equations, we can see that the following relationship holds in the steady state:

\[
k_{g} = \lambda \frac{s^g_{pr}}{s^f_{pr}} k_{pr}
\]
(45),

which will be called steady state locus (SSL) in the following. Along the SSL the percentage growth rates of the private and the public capital-to-labor ratios are equal to each other. The SSL value of the ratio of private to public capital, defined as:

\[
\kappa_g^{pr} = \frac{\kappa_{pr}}{\kappa_g}
\]
(46),

is:

\[
\left( \kappa_g^{pr} \right)_{SSL} = \frac{s^f_{pr}}{\lambda s^g_{pr}}
\]
(47).

With eqs (43) and (44) we can derive the corresponding differential equation

\[
\dot{k}_g^{pr} = \left( \dot{k}_{pr} - \dot{k}_g \right) k_g^{pr} = (1 - \tau) \gamma \left[ 1 + \lambda^{-1}(1 - \sigma)(1 - \alpha) \left( \kappa_g^{pr} \right)^{-1} \right] \kappa_g^{-\alpha} \left( \kappa_g^{pr} \right)^{\beta} \left( \frac{s^f_{pr} - \lambda s^g_{pr} k_{pr}^{pr}}{s^g_{pr}} \right)
\]
(48).

For a given public capital-to-labor ratio we can derive the result that the slope of the differential equation is negative on the SSL:

\[
\frac{\partial k_g^{pr}}{\partial k_{g}^{pr}} = -\lambda s^g_{pr} (1 - \tau) \gamma \left[ 1 + (1 - \sigma)(1 - \alpha) \left( \frac{s^f_{pr}}{s^g_{pr}} \right)^{-1} \right] \left( \frac{s^f_{pr}}{\lambda s^g_{pr} \kappa_g} \right)^\beta < 0
\]
(49).
Hence, for a given public capital-to-labor ratio the SSL is at least locally asymptotically stable. In the steady state the growth rates of national income, private and public capital as well as public debt coincide with the growth rate of labor supply in efficiency units:

\[ \dot{Y}^* = \dot{K}^*_p = \dot{K}^*_g = \dot{B}^* = \delta + n \]  

(50).

Englmann 2016 shows that the steady state is unique and stable if the safety discount factor and the propensities to invest are exogenously given and constant, and as long as the growth rate of labor supply in efficiency units \((\delta + n)\) and the production elasticity of labor are positive. In order to satisfy the intertemporal budget constraint (or solvency constraint) in the steady state, the rate of interest on government bonds has to exceed the steady state growth rate of national income. Whether the intertemporal budget constraint is fulfilled depends on the absolute values of the tax rate, the propensities to invest, the production elasticities of private and public capital and the safety discount factor.

**Figure 1: Exogenous safety discount factor: combinations of tax rate and propensity to invest in government bonds for which the steady state debt ratio is long-term sustainable and efficient**

\[
\begin{align*}
&\text{(a) Constant discount factor: } \sigma = 0.0 \\
&\text{(b) Constant discount factor: } \sigma = 0.7 \\
&\text{(c) Constant discount factor: } \sigma = 0.8 \\
&\text{(d) Constant discount factor: } \sigma = 1.
\end{align*}
\]

For various safety discount factors, the gray areas in Figure 1 show combinations of the tax rate and the propensity to invest in government bonds for which the respective steady state debt ratio is efficient and long-term sustainable in the sense that both the liquidity and the solvency constraints are satisfied in the stable steady state.

Following Englmann 2016 we can derive the minimum threshold of the tax rate \(\tau_{\text{Min}}^{s_{\text{solv}}}\) for values above which the government’s solvency constraint is satisfied:

\[ \tau \geq \tau_{\text{Min}}^{s_{\text{solv}}} = 1 - \frac{(1 - \sigma)(1 - \alpha)}{s_{pr} + s_{pr}(1 - \sigma)(1 - \alpha)}; \quad \frac{\partial \tau_{\text{Min}}^{s_{\text{solv}}}}{\partial \sigma} > 0 \]  

(51),

as well as the minimum tax rate for which the liquidity constraint is fulfilled:
and the maximum propensity to invest in government bonds that ensures efficiency in the sense that the shadow rate of return on the public budget deficit is not smaller than the steady state rate of interest on government bonds

$$\left( s_{pr}^g \right)_{\text{Max}}^{\text{eff}} = \frac{s_{pr}^f (1 - \alpha - \beta)}{(1 - \sigma)(1 - \alpha)}; \quad \frac{\partial \left( s_{pr}^g \right)_{\text{Max}}^{\text{eff}}}{\partial \sigma} > 0$$

(53).

By comparing the various graphs in Figure 1 we see that the larger is the safety discount factor, the larger are $$\tau_{\text{Min}}$$ and $$\left( s_{pr}^g \right)_{\text{Max}}^{\text{eff}}$$, and the smaller is $$\tau_{\text{Min}}^{\text{liq}}$$. But stability of the steady state is not affected by the safety discount factor and hence by the rate of interest, as long as the safety discount factor is exogenous and hence as long as the rate of interest decreases with the debt ratio, at least along the stable SSL. Yet, given the evidence reported e.g. in Greenlaw et al. 2013 and the literature referred to therein, it is reasonable to assume that investors demand a higher rate of return on government bonds relative to private capital once government bonds are considered less safe due to increasing debt-to-GDP ratios. Hence, in the following we will assume that the safety discount factor decreases with an increase in the debt ratio.

3. Endogenous safety discount factor, exogenous propensity to invest in government bonds, and exogenous tax rate

In this section only the safety discount factor will be endogenized, whereas the propensities to invest and the tax rate remain exogenously given and constant. We stipulate that the safety discount factor decreases with a rise in the public debt ratio within certain limits:

$$\sigma = \begin{cases} \bar{\sigma} & \text{if } b \leq 0 \\ \sigma(b); & \sigma’(b) < 0 \text{ if } b > 0 \end{cases}$$

(54).

With eqs. (54), (35), and (49) the slope of the differential equation for the private-to-public-capital ratio on the SSL remains negative because of $$\sigma(b) \leq 1$$:

\[
\frac{\partial \kappa_{pr}}{\partial \kappa_{g}} = -\lambda s_{pr}^g (1 - \tau) \left\{ 1 + \left[ 1 - \sigma \left[ \lambda^{-1} \gamma^{-1} \left( \frac{s_{pr}^f}{s_{pr}^g} \right)^{-\beta} \frac{\kappa_{g}^{\alpha}}{\kappa_{g}} \right] \right] \left[ (1 - \alpha) \left( \frac{s_{pr}^f}{s_{pr}^g} \right)^{-1} \right] \right\} \lambda s_{pr}^g \frac{\beta}{\kappa_{g}^{-\alpha}} < 0
\]

(55).

Hence, for a given public capital-to-labor ratio the SSL remains at least locally asymptotically stable with a varying safety discount factor as long as the propensities to invest remain constant.

As shown in Englmann 2016 the following differential equation for the debt-to-GDP ratio can easily be derived:

$$\dot{b} = d - (\dot{y} - fb)b$$

(56), and hence with eq. (15)

$$\dot{b} = s_{pr}^g (1 - \tau) (1 + fb) - \dot{y}b$$

(57).
The GDP growth rate, the rate of interest on government bonds, and the debt ratio as well as their changes in time are functions of the private and public capital-to-labor ratios, but the GDP growth rate and the rate of interest on government bonds are not (direct) functions of the debt ratio. Hence, in order to make the GDP growth rate and the rate of interest on government bonds direct functions of the debt ratio we have to consider the development along specific paths in the \((\kappa_{pr}, \kappa_{g})\) phase diagram, i.e. for specific combinations of private and the public capital-to-labor ratios.

There are various alternatives that might be chosen like the \(\kappa_{pr} = 0\) - isocline, the \(\kappa_{g} = 0\) - isocline, and the steady state locus. There are two reasons why we will concentrate on the SSL: (i) only in case of the SSL the private capital-to-labor ratio can be expressed analytically as a function of the public capital-to-labor ratio, and (ii) the SSL is the only one of the three loci that is not crossed by a trajectory (see Figure 5 (a), (c), (e), and (g)), as long as the propensities to invest are exogenously given. In this case the steady state locus (SSL) is given by eq. (45) and hence a straight line in the \(\kappa_{pr} - \kappa_{g}\)-phase diagram. If the initial values of the private and the public capital-to-labor ratios are located on the SSL they will stay on the SSL, as the percentage rates of change of the private and public capital-to-labor ratios are identical. Hence, the SSL is a balanced growth path of private and public capital. If the initial values of the private and the public capital-to-labor ratios are not located on the SSL, the SSL will be approached asymptotically.

Along the SSL the rate of return on private capital falls with the debt ratio, as we can derive from eqs (35), (42), and (45):

\[
\eta_{SSL}^{K} = \frac{s_{pr}^{g} (1 - \alpha)}{s_{pr}^{f}} (b_{SSL})^{-1}
\]

Hence, according to eq. (29) we get for the rate of interest on government bonds along the SSL:

\[
\eta_{SSL}^{B} = \frac{s_{pr}^{g} \left[ 1 - \sigma \left( b_{SSL} \right) \right] (1 - \alpha)}{s_{pr}^{f}} (b_{SSL})^{-1}
\]

If the safety discount factor is constant, along the SSL the rate of interest on government bonds decreases when the debt ratio increases. Given the evidence reported, e.g., in Laubach 2009 as well as in Greenlaw et al 2013 and the literature referred to therein, this is a rather unrealistic assumption, as, in general the rate of interest on government bonds rises relative to the rate of return on private capital if the public debt ratio increases.

In order to investigate whether the functional form of the safety discount factor plays a role with respect to existence, uniqueness, and stability of the steady state we will stipulate various functional relationships, namely square root, linear, quadratic, and cubic (see eqs (60) – (63)).

\[
\sigma^r = \overline{\sigma} - \omega \sqrt{b}; \quad \omega > 0
\]

\[
\sigma^l = \overline{\sigma} - \omega b; \quad \omega > 0
\]

\[
\sigma^q = \overline{\sigma} - \omega b^2; \quad \omega > 0
\]

\[
\sigma^c = \overline{\sigma} - \omega b^3; \quad \omega > 0
\]

or more compactly:
\[ \sigma^z = \sigma - \omega b^m; \quad \omega > 0; \quad z = r, l, q, c; \quad m = 0.5, 1, 2, 3 \]  \hspace{1cm} (64)\,^5

Eq. (60) implies that the rate of return on an investment in government bonds relative to the rate of return on private capital that investors require is the higher, the higher is the debt ratio as investors consider government bonds less safe. We assume that the upper threshold of the safety discount factor equals one (see Figure 2 (a)).

(\sigma = 1, \omega = 1)

From eqs (33), (35), and (45) we can deduce the following positive relation between per capita income and the debt ratio along the SSL:

\[ y^{SSL} = y^{1/\alpha} l^{\alpha/(1-\alpha)} \left( s^{f}_{pr} \right)^{b/\alpha} \left( b^{SSL} \right)^{(1-\alpha)/\alpha} \]  \hspace{1cm} (65).

For the percentage rate of change of national income we obtain from eqs (33), (1), (30), and (31):

\[ \dot{Y}^{SSL} = (1-\alpha) \dot{k}_g^{SSL} + (\delta + n) \]  \hspace{1cm} (66),

and for the percentage rate of change of the public-capital-to-labor ratio from eqs (44) and (59) along the SSL, where the index \( \alpha^{SSL} \) refers to the case of a variable safety discount factor depending on the debt ratio:

\[ \dot{k}_g^{\alpha^{SSL}} = s^{g}_{pr} (1-\tau) \left\{ 1 + (1-\alpha) \frac{s^{g}_{pr}}{s^{f}_{pr}} \left[ 1 - \sigma \left( b^{\alpha^{SSL}} \right) \right] \right\} \frac{1}{b^{\alpha^{SSL}}} - (\delta + n) \]  \hspace{1cm} (67).

By inserting eq. (67) into eq. (66) we derive the following relationship between the growth rate of national income and the debt ratio along the SSL:

\[ \dot{Y}^{\alpha^{SSL}} = (1-\alpha) \dot{s}^{\alpha^{SSL}}_{pr} \left( b^{\alpha^{SSL}} \right) \frac{1}{b^{\alpha^{SSL}}} + \alpha(\delta + n) \]  \hspace{1cm} (68),

where

\[ \dot{s}^{\alpha^{SSL}}_{pr} \left( b^{\alpha^{SSL}} \right) = s^{g}_{pr} (1-\tau) \left\{ 1 + (1-\alpha) \frac{s^{g}_{pr}}{s^{f}_{pr}} \left[ 1 - \sigma \left( b^{\alpha^{SSL}} \right) \right] \right\} \dot{B} \left( b^{\alpha^{SSL}} \right) \]  \hspace{1cm} (69).

With eq. (57) we finally get the differential equation for the debt ratio along the SSL:

\[ \dot{k}_g^{\alpha^{SSL}} = s^{g}_{pr} (1-\tau) \left\{ 1 + (1-\alpha) \frac{s^{g}_{pr}}{s^{f}_{pr}} \left[ 1 - \sigma \left( b^{\alpha^{SSL}} \right) \right] \right\} \frac{1}{b^{\alpha^{SSL}}} - (\delta + n) \]  \hspace{1cm} (67).

\[ \dot{Y}^{\alpha^{SSL}} = (1-\alpha) \dot{s}^{\alpha^{SSL}}_{pr} \left( b^{\alpha^{SSL}} \right) \frac{1}{b^{\alpha^{SSL}}} + \alpha(\delta + n) \]  \hspace{1cm} (68),

where

\[ \dot{s}^{\alpha^{SSL}}_{pr} \left( b^{\alpha^{SSL}} \right) = s^{g}_{pr} (1-\tau) \left\{ 1 + (1-\alpha) \frac{s^{g}_{pr}}{s^{f}_{pr}} \left[ 1 - \sigma \left( b^{\alpha^{SSL}} \right) \right] \right\} \dot{B} \left( b^{\alpha^{SSL}} \right) \]  \hspace{1cm} (69).

With eq. (57) we finally get the differential equation for the debt ratio along the SSL:

\[ \dot{k}_g^{\alpha^{SSL}} = s^{g}_{pr} (1-\tau) \left\{ 1 + (1-\alpha) \frac{s^{g}_{pr}}{s^{f}_{pr}} \left[ 1 - \sigma \left( b^{\alpha^{SSL}} \right) \right] \right\} \frac{1}{b^{\alpha^{SSL}}} - (\delta + n) \]  \hspace{1cm} (67).

\[ \dot{Y}^{\alpha^{SSL}} = (1-\alpha) \dot{s}^{\alpha^{SSL}}_{pr} \left( b^{\alpha^{SSL}} \right) \frac{1}{b^{\alpha^{SSL}}} + \alpha(\delta + n) \]  \hspace{1cm} (68),

where

\[ \dot{s}^{\alpha^{SSL}}_{pr} \left( b^{\alpha^{SSL}} \right) = s^{g}_{pr} (1-\tau) \left\{ 1 + (1-\alpha) \frac{s^{g}_{pr}}{s^{f}_{pr}} \left[ 1 - \sigma \left( b^{\alpha^{SSL}} \right) \right] \right\} \dot{B} \left( b^{\alpha^{SSL}} \right) \]  \hspace{1cm} (69).

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\[ \dot{Y}^{\alpha^{SSL}} = (1-\alpha) \dot{s}^{\alpha^{SSL}}_{pr} \left( b^{\alpha^{SSL}} \right) \frac{1}{b^{\alpha^{SSL}}} + \alpha(\delta + n) \]  \hspace{1cm} (68),

where

\[ \dot{s}^{\alpha^{SSL}}_{pr} \left( b^{\alpha^{SSL}} \right) = s^{g}_{pr} (1-\tau) \left\{ 1 + (1-\alpha) \frac{s^{g}_{pr}}{s^{f}_{pr}} \left[ 1 - \sigma \left( b^{\alpha^{SSL}} \right) \right] \right\} \dot{B} \left( b^{\alpha^{SSL}} \right) \]  \hspace{1cm} (69).

\[ \dot{k}_g^{\alpha^{SSL}} = s^{g}_{pr} (1-\tau) \left\{ 1 + (1-\alpha) \frac{s^{g}_{pr}}{s^{f}_{pr}} \left[ 1 - \sigma \left( b^{\alpha^{SSL}} \right) \right] \right\} \frac{1}{b^{\alpha^{SSL}}} - (\delta + n) \]  \hspace{1cm} (67).

\[ \dot{Y}^{\alpha^{SSL}} = (1-\alpha) \dot{s}^{\alpha^{SSL}}_{pr} \left( b^{\alpha^{SSL}} \right) \frac{1}{b^{\alpha^{SSL}}} + \alpha(\delta + n) \]  \hspace{1cm} (68),

where

\[ \dot{s}^{\alpha^{SSL}}_{pr} \left( b^{\alpha^{SSL}} \right) = s^{g}_{pr} (1-\tau) \left\{ 1 + (1-\alpha) \frac{s^{g}_{pr}}{s^{f}_{pr}} \left[ 1 - \sigma \left( b^{\alpha^{SSL}} \right) \right] \right\} \dot{B} \left( b^{\alpha^{SSL}} \right) \]  \hspace{1cm} (69).
\( b^\sigma \text{SSL} = \alpha \frac{\hat{B}}{V}(b^\sigma \text{SSL}) - \alpha(\sigma + n)b^\sigma \text{SSL} \quad (70), \)

or with eq. (64) and \( \overline{\sigma} = 1: \)

\[
\frac{\partial b^\sigma \text{SSL}}{\partial b^\sigma \text{SSL}} (b^\sigma^*) = -\alpha(1-\tau)(1-\alpha) \left( \frac{s_{pr}^g}{s_{pr}^f} \right)^2 \omega \left( b^{\sigma*} \right)^{m-1} \alpha(\sigma + n) ; \quad z = r, l, q, c; \quad m = 0.5, 1, 2, 3 \]

whose steady state slopes are:

\[
\frac{\partial \dot{b}^\sigma \text{SSL}}{\partial b^\sigma \text{SSL}} (b^\sigma^*) = -\alpha(1-\tau)(1-\alpha) \left( \frac{s_{pr}^g}{s_{pr}^f} \right)^2 \omega \left( b^{\sigma*} \right)^{m-1} \alpha(\sigma + n) \quad (72),
\]

or with eq. (64) and \( \overline{\sigma} = 1: \)

\[
\frac{\partial \dot{b}^\sigma \text{SSL}}{\partial b^\sigma \text{SSL}} (b^{\sigma*}) = \alpha(1-\tau)(1-\alpha) \left( \frac{s_{pr}^g}{s_{pr}^f} \right)^2 \omega m \left( b^{\sigma*} \right)^{m-1} \alpha(\sigma + n) ; \quad z = r, l, q, c; \quad m = 0.5, 1, 2, 3 \]

Depending on the exponent in the safety discount factor function and the values of the other parameters there are up to three steady states. For \( m = 0.5 \) there is a maximum number of two steady states, the larger of which is stable; for \( m = 1 \) there is a maximum number of one steady state which is stable or unstable;\(^6\) for \( m = 2 \) there is a maximum number of two steady states, the smaller of which is stable; for \( m = 3 \) there is a maximum number of three steady states, of which the middle one is stable. A steady state is stable if the phase curve of the debt ratio is negatively sloped, hence if

\[
\sigma'(b^{\sigma*}) > \frac{(\sigma + n)s_{pr}^f}{(1-\tau)(1-\alpha) \left( \frac{s_{pr}^g}{s_{pr}^f} \right)^2} \quad (74),
\]

or with eq. (64) and \( \overline{\sigma} = 1: \)

\[
b^{\sigma*} < (m-1) \sqrt{\left( \frac{(\sigma + n)s_{pr}^f}{(1-\tau)(1-\alpha) \omega \left( \frac{s_{pr}^g}{s_{pr}^f} \right)^2} \right) ; \quad z = r, l, q, c; \quad m = 0.5, 1, 2, 3 \quad (75),
\]

i.e. if the safety discount factor does not decrease too strongly with an increase in the debt ratio (see Figure 5) or if the tax rate exceeds a minimum threshold:

\[
\tau > \tau_{\text{Min}}^{\text{sta}} = 1 + \frac{(\sigma + n)s_{pr}^f}{\sigma'(b^{\sigma*})(1-\alpha) \left( \frac{s_{pr}^g}{s_{pr}^f} \right)^2} ; \quad \sigma'(b^{\sigma*}) < 0 \quad (76),
\]

or with eq. (64) and \( \overline{\sigma} = 1: \)

\(^6\) In order to yield an unstable steady state for the parameter values in Figure 5, the first derivative of the safety discount factor with respect to the debt ratio would have to be smaller than –23.8095, i.e. an increase in the debt ratio by one percentage point would have to lower the safety discount factor by more than 23.8095, which is rather unrealistic.
\[
\tau > \tau_{\text{Min}}^{\text{sta}} = 1 - \frac{(\delta + n)s_{pr}^{f}}{\omega m (b^{\alpha z^*})^{m-1} (1 - \alpha) \left( s_{pr}^{g} \right)^{m}}; \quad z = r, l, q, c; \quad m = 0.5, 1, 2, 3 \quad (77),
\]

where the steady state value of the debt ratio depends on the tax rate, too. This may make an analytical solution of the following equation difficult or even impossible if \( m \) exceeds four:

\[
\tau > \tau_{\text{Min}}^{\text{sta}} = \text{Solve} \left( b^{\alpha z^*} = (m-1) \left( \frac{(\delta + n)s_{pr}^{f}}{s_{pr}^{g}} \right)^{1/(1 - \alpha)m} \right); \quad z = r, l, q, c; \quad m = 0.5, 1, 2, 3 \quad (78).
\]

Figure 3 shows the respective minimum tax rates for stable steady state debt ratios. The higher the exponent \( m \) of the safety discount factor function, the higher the tax rate has to be in order to stabilize the steady state.

If the safety discount factor function depends on the square root of the debt ratio we can derive for the rate of interest along the SSL:

\[
r_{B}^{\text{critSSL}} = (1 - \alpha) \frac{s_{pr}^{g}}{s_{pr}^{pr}} \frac{1}{\sqrt{b^{\alpha rSSL}}} \quad \frac{\partial r_{B}^{\text{critSSL}}}{\partial \omega} > 0, \quad \frac{\partial r_{B}^{\text{critSSL}}}{\partial b^{\alpha rSSL}} < 0 \quad (79).
\]

In case of a linear safety discount factor function (see eq. (61)), i.e. if the safety discount factor decreases linearly with the debt ratio, the rate of interest is constant along the SSL and hence equal to its steady state value:

\[
r_{B}^{r} = r_{B}^{\text{critSSL}} = (1 - \alpha) \frac{s_{pr}^{g}}{s_{pr}^{pr}} \omega; \quad \frac{\partial r_{B}^{r}}{\partial \omega} > 0, \quad \frac{\partial r_{B}^{r}}{\partial b_{o}^{\text{critSSL}}} = 0 \quad (80).
\]

According to eq. (80) the steady state rate of interest is the higher, the higher the production elasticity of capital, the higher the propensity to invest in government bonds, and the more the safety discount factor falls with the debt ratio. In case of a quadratic safety discount factor function the rate of interest is a linear function of the debt ratio along the SSL:

\[
r_{B}^{rSSL} = (1 - \alpha) \frac{s_{pr}^{g}}{s_{pr}^{pr}} \omega b_{o}^{\text{critSSL}}; \quad \frac{\partial r_{B}^{rSSL}}{\partial \omega} > 0, \quad \frac{\partial r_{B}^{rSSL}}{\partial b_{o}^{\text{critSSL}}} > 0 \quad (81),
\]
in case of a cubic safety discount factor function the rate of interest is a quadratic function of the debt ratio along the SSL:

\[ r^\delta_{SSL} = (1 - \alpha) \frac{s^g_{pr}}{s^I_{pr}} \omega \left( b^\delta_{SSL} \right)^2; \quad \frac{\partial r^\delta_{SSL}}{\partial \omega} > 0, \quad \frac{\partial^2 r^\delta_{SSL}}{\partial \omega^2} > 0 \]  

(82).

With eq. (64) and \( \sigma = 1 \) eqs (79) - (82) can be written in a compact way as follows:

\[ r^\delta_{SSL} = (1 - \alpha) \frac{s^g_{pr}}{s^I_{pr}} \omega \left( b^\delta_{SSL} \right)^{m - 1}; \quad \frac{\partial r^\delta_{SSL}}{\partial \omega} > 0; \quad z = r, l, q, c; \quad m = 0.5, 1, 2, 3 \]  

(83).

From eqs (79) - (82) as well as from Figure 2 (b) we see that, for high debt ratios in absolute terms the rate of interest on government bonds only increases with the debt ratio along the SSL if the safety discount function is over-linear (\( m > 1 \)). This is due to decreasing returns on private capital. Hence, in order to obtain an increase in the interest rate in case of a rise in the debt ratio along the SSL the safety discount factor function has to be over-linear. The corresponding steady state values of the debt ratio can be computed by setting eq. (70) equal to zero and taking eq. (80) into account (see eqs (111) - (114) in the appendix).

From eq. (112) we see that the steady state debt ratio increases with the absolute value of the marginal safety discount factor rate \( \omega \).  

By using eq. (80) we can rewrite the condition for steady state stability (74) in the following way:

\[ r^*_{cr} < \frac{2(\delta + n)}{(1 - \tau)s^{g}_{pr}} \]  

(84).

\[ r^*_{ci} = r^\delta_{SSL} < \frac{2(\delta + n)}{(1 - \tau)s^{g}_{pr}} \]  

(85).

\[ r^*_{cq} < \frac{2(\delta + n)}{(1 - \tau)s^{g}_{pr}} \]  

(86).

\[ r^*_{cb} < \frac{2(\delta + n)}{(1 - \tau)s^{g}_{pr}} \]  

(87).

or with eq. (64) and \( \sigma = 1 \) in a more compact way:

\[ r^*_{cz} < \frac{(\delta + n)}{m(1 - \tau)s^{g}_{pr}}; \quad z = r, l, q, c; \quad m = 0.5, 1, 2, 3 \]  

(88).

Hence, the higher the exponent \( m \) in the safety discount factor function, the lower the steady state interest rate has to be in order to stabilize the steady state. This may lead to a conflict with the government’s solvency. In order to satisfy the solvency constraint the steady state interest rate has to exceed the steady state growth rate:

\[ r^*_{cz} > \delta + n; \quad z = r, l, q, c; \quad m = 0.5, 1, 2, 3 \]  

(89).

\[ ^7 \text{Hence, the impact of an increase in the absolute value of the marginal safety discount factor rate, which decreases the safety discount factor for any given debt ratio, on the steady state rate of interest and the steady state debt ratio is analogous to a decrease in the safety discount factor if the latter is exogenous.} \]
The corresponding maximum tax rate, for values strictly below which condition (89) is satisfied, can be computed analytically, at least as long as $0 \leq m \leq 2$:

$$
\tau < \tau^\text{sol}_\text{Max} = \text{Solve} \left\{ b^{\alpha z^*} = \left( m-1 \right) \sqrt{\frac{(\delta + n)s^f_{pr}}{\omega(1-\alpha)s^g_{pr}}} \tau \right\}; \ z = r, l, q, c; \ m = 0.5, 1, 2, 3 \quad (90).
$$

Obviously, conditions (88) and (89) can be satisfied simultaneously for

$$
\tau^\alpha z > 0 > -1 + m s^g_{pr} \quad ; \ z = r, l, q, c; \ m = 0.5, 1, 2, 3 \quad (91).
$$

In general, condition (91) is fulfilled. Hence, the cases with a fixed tax rate do not necessarily lead to a conflict between stability and solvency. Now let us turn to the question of liquidity. In the models presented here public consumption is determined endogenously for a given tax rate and a given public-investment-to-budget-deficit ratio, which are both determined by the government (see (22)). For the government to be liquid, public consumption has to be nonnegative. Along the SSL we can derive from eqs (22) and (83) for the share of public consumption in national income:

$$
\left( \frac{C^g}{Y} \right)^{\alpha z^{SSL}} = 1 - \left( 1 - (1-\lambda)s^g_{pr} \right)(1-\tau) \left[ 1 + (1-\alpha) \frac{s^g_{pr}}{s^f_{pr}} \omega \left( b^{\alpha z^{SSL}} \right)^m \right] ; \ z = r, l, q, c; \ m = 0.5, 1, 2, 3 \quad (92),
$$

which is positive for

$$
\tau > \tau^{\text{liq}}^{\alpha z^*} = 1 - \frac{s^f_{pr}}{\left( 1 - (1-\lambda)s^g_{pr} \right) \left[ \frac{f}{s^f_{pr} + \omega \left( b^{\alpha z^*} \right)^m (1-\alpha)s^g_{pr}} \right]} ; \ z = r, l, q, c; \ m = 0.5, 1, 2, 3 \quad (93).
$$

$\tau^{\text{liq}}^{\alpha z^*}$ can be determined analytically for $m = 0, 0.5, 1, 2$ and numerically at least for $m = 3$ by solving the following equations:

$$
\tau > \tau^{\text{Min}}^{\text{liq}} = \text{Solve} \left\{ b^{\alpha z^*} = \left[ \frac{s^f_{pr} (1-\lambda) (1-\tau)s^g_{pr} + \tau}{\omega (1-\alpha) (1-\lambda)s^g_{pr} (1-\lambda)s^g_{pr}} \right]^m \tau \right\}; \ z = r, l, q, c; \ m = 0.5, 1, 2, 3 \quad (94),
$$

Finally, we can determine the maximum tax rate for which steady state public debt is efficient, i.e. for which holds:

$$
\tau^*_B \leq \tau^{\alpha z^*}; \ z = r, l, q, c; \ m = 0.5, 1, 2, 3 \quad (95).
$$

The maximum tax rate that ensures condition (95) to be fulfilled is determined by the following equations:

$$
\tau < \tau^{\text{Min}}^{\text{eff}} = \text{Solve} \left\{ b^{\alpha z^*} = \left[ \frac{(1-\alpha-\beta)s^f_{pr}}{\omega (1-\alpha)s^g_{pr}} \right], \tau \right\}; \ z = r, l, q, c; \ m = 0.5, 1, 2, 3 \quad (96),
$$

Figure 4 shows various sets of combinations of the propensity to invest in government bonds and the tax rate by differently colored areas as specified in Table 2. For combinations in the light gray areas steady
state public debt is both long-term sustainable and efficient, for combinations in the light blue areas steady state public debt is long-term sustainable yet inefficiently high. By comparing the graphs of Figure 4 (c) and (d) as well (e) and (f) we see that the set of combinations of the propensity to invest in government bonds and the tax rate for which steady state public debt is long-term sustainable shrinks if the rate of interest on government bonds reacts more strongly to changes in the debt ratio. Hence, if Central Banks intervene in the secondary market for government bonds, as they have done since the outbreak of the Great Recession, and thus lower the marginal safety factor discount rate this does not only impact the liquidity of governments, but their solvency and the stability of public debt as well. For a given propensity to invest in government bonds public debt can be long-term sustainable with lower tax rates.

Figure 5 shows stream plots for various cases as well as corresponding diagrams of the dynamics along the SSL. Figure 5 (e) and (f) show a case where a positive steady state does not exist, as the combination
of the propensity to invest in government bonds and the tax rate is located neither in the light gray nor the light blue area. Furthermore, Figure 5 (e) and (f) also highlight the fact that stability and existence of a steady state are intimately related. The slope of the phase curve of the debt ratio becomes zero when the former just touches the abscissa. Hence, once the steady state becomes unstable, it even ceases to exist and the debt ratio increases without bound.

This is clearly a situation where austerity measures would be discussed. In the Eurozone there are two approaches that are used: (i) a budget deficit-to-GDP ratio target and (ii) a primary surplus-to-GDP ratio target. We will implement these two approaches by assuming that the tax rate will be adjusted in such a way as to realize either the former or the latter. As a primary surplus is a necessary condition to satisfy the government’s intertemporal budget constraint, at first sight this seems to be the better choice. But we will see in the next section that this does not solve the problem of instability.

<table>
<thead>
<tr>
<th>Areas in Figure 4</th>
<th>Government’s liquidity constraint</th>
<th>Stability conditions</th>
<th>Government’s solvency constraint</th>
<th>Efficiency condition</th>
<th>Sustain-ability and efficiency</th>
</tr>
</thead>
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<tr>
<td>Light green</td>
<td>violated in the steady state</td>
<td>violated in the steady state</td>
<td>satisfied in the steady state</td>
<td>violated in the steady state</td>
<td>short and medium-term unsustainable and inefficient</td>
</tr>
<tr>
<td>Light red</td>
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<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
<td>satisfied or violated in the steady state</td>
<td>short-term unsustainable yet medium- and long-term sustainable</td>
</tr>
<tr>
<td>Light blue</td>
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<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
<td>violated in the steady state</td>
<td>long-term sustainable yet inefficient</td>
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<tr>
<td>Light gray</td>
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<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
<td>long-term sustainable and efficient</td>
</tr>
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Table 2: Sets of combinations of exogenous propensity to invest in government bonds and exogenous tax rate

4. Budget Deficit Ratio Target versus Primary Surplus Ratio Target

In this section we want to investigate the debt dynamics that are involved by targeting either a budget deficit ratio or a primary surplus ratio. In the model presented here, the policy instrument to be used for either target is the tax rate. In order to target a specific budget deficit ratio \( \mu \), the tax rate has to be determined in such a way that the actual budget deficit ratio just equals the target ratio (see Englmann 2016):

\[
\frac{\dot{B}}{Y} = \frac{g_{pr}}{Y} = \frac{g_{pr}}{pr} (1 - \tau^\mu) \left(1 + \frac{s_{pr}^b}{\tau^\mu} b^\mu\right) = \mu \tag{97}
\]

From eq. (97) we can immediately derive the required tax rate \( \tau^\mu \):

\[
\tau^\mu = 1 - \frac{\mu}{g_{pr} \left(1 + \frac{s_{pr}^b}{\tau^\mu} b^\mu\right)}; \quad 0 < \tau^\mu < 1; \quad \frac{\partial \tau^\mu}{\partial b^\mu} = \frac{\mu}{s_{pr}^b \left(1 + \frac{s_{pr}^b}{\tau^\mu} b^\mu\right)^2} > 0 \tag{98}
\]
Figure 5: Fixing the tax rate: stream plots and debt ratio phase curves

(a) Stream plot: square root safety discount factor function
(b) Phase curve of the debt ratio: square root safety discount factor function
(c) Stream plot: linear safety discount factor function
(d) Phase curve of the debt ratio: linear safety discount factor function
(e) Stream plot (s_0^{pr} = 0.02): cubic safety discount factor function
(f) Phase curve of the debt ratio (s_0^{pr} = 0.02): cubic safety discount factor function
(g) Stream plot (s_0^{pr} = 0.035): cubic safety discount factor function
(h) Phase curve of the debt ratio (s_0^{pr} = 0.035): cubic safety discount factor function

Figure 5: Fixing the tax rate: stream plots and debt ratio phase curves

(s_0^{pr} = 0.7, \gamma = 0.2, \alpha = 0.7, \beta = 0.2, \tau = 0.3, n = 0, \delta = 0.02, \lambda = 0.01, \gamma = 0.01)
The more the rate of interest rises with the debt ratio, the more the tax rate increases in order to keep the budget deficit ratio constant. By inserting eq. (98) into the system of differential equations (43) and (44) we obtain the following system of differential equations:

\[
\dot{\kappa}^\mu = \frac{s}{g} \mu \gamma \left( \kappa^\mu \right)^{1-\alpha} - (\delta + n) \kappa^\mu
\]

(99),

\[
\dot{\kappa}^\sigma = \lambda \mu \gamma \left( \kappa^\mu \right)^{1-\alpha} - (\delta + n) \kappa^\sigma
\]

(100).

Accordingly, the dynamics of private and public capital as well as of public debt do not depend on the rate of interest or safety discount factor function. The SSL remains unchanged and stable. Hence, again, public debt dynamics can be well approximated by movements along the SSL. With a budget deficit ratio target the differential equation for the debt ratio becomes:

\[
\dot{b}^\mu = \alpha \mu - \alpha (\delta + n) b^\mu
\]

(101).

Independent of the safety discount factor function the steady state is unique and stable if \(\alpha, \mu, (\delta + n) > 0\). The steady state debt ratio is:

\[
b^\mu = \frac{\mu}{\delta + n}
\]

(102).

Figure 6: Sustainable combinations of budget deficit ratio target and propensity to invest in government bonds

Even if the steady state debt ratio does not depend on the safety discount factor function the corresponding interest rate does. The gray areas in Figure 6 show the set of combinations of the budget deficit target ratio and the propensity to invest in government bonds for which public debt is long-term sustainable and efficient. By comparing the various graphs we observe that the sets of long-term sustainable and efficient
combinations are the smaller, the more the rate of interest varies with the debt ratio, i.e. the more sensitive investors in government bonds become with respect to the debt ratio.

<table>
<thead>
<tr>
<th>Areas in Figure 6 and Figure 8</th>
<th>Government’s liquidity constraint</th>
<th>Stability conditions</th>
<th>Government’s solvency constraint</th>
<th>Efficiency condition</th>
<th>Sustainability and efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Light red</td>
<td>violated in the steady state</td>
<td>satisfied in the steady state</td>
<td>satisfied or violated in the steady state</td>
<td>satisfied or violated in the steady state</td>
<td>short-term unsustainable</td>
</tr>
<tr>
<td>White</td>
<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
<td>violated in the steady state</td>
<td>satisfied in the steady state</td>
<td>long-term unsustainable</td>
</tr>
<tr>
<td>Light blue</td>
<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
<td>violated in the steady state</td>
<td>long-term sustainable yet inefficient</td>
</tr>
<tr>
<td>Light gray</td>
<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
<td>long-term sustainable and efficient</td>
</tr>
</tbody>
</table>

Table 3: Sets of combinations of exogenous propensity to invest in government bonds and budget deficit ratio target

Analytical computations as well as Figure 7 show that fixing the tax rate and targeting a specific budget deficit ratio lead to identical sustainability and efficiency thresholds with respect to the implied steady state primary surplus ratios except stability. In case of a fixed tax rate stability depends on both the tax rate and the propensity to invest in government bonds (see $-\sigma^* \left( \frac{s^g_{Sol}}{s^g_{Min}} \right)^{\sigma_q}$ in Figure 7, but not in case of a budget deficit ratio target.

Figure 7: Budget deficit ratio target and fixed tax rate compared: sustainable combinations of steady state primary surplus ratio and propensity to invest in government bonds

It is noteworthy that in case of a quadratic or cubic safety factor discount function a complete debt brake or Dr. Schäuble’s ‘black zero’ is not sustainable as it jeopardizes the government’s solvency due to the rate of interest on government bonds being smaller than the growth rate of national income. As long as the propensity to invest in government bonds is positive a debt brake leading to a zero debt ratio in the long run is not in line with private households’ preferences.
Figure 8 shows the corresponding sustainable combinations of steady state primary surplus ratio and propensity to invest in government bonds. We see that the government can indirectly target a sustainable and efficient primary surplus ratio via targeting a sustainable and efficient budget deficit ratio.

Yet, within the Euro area there has been a tendency to move away from a budget deficit ratio target since the European debt crisis. Whereas a budget-deficit-to-GDP ratio is targeted in the Maastricht Treaty and in the Growth and Stability Pact, a primary surplus-to-GDP ratio ($\pi$) is targeted in the Treaty on Stability Coordination and Governance concerning the adjustment programmes for Greece, Ireland, Portugal, and Spain (see European Commission 2013). If the primary surplus ratio is successfully targeted the budget deficit ratio takes the following form:

$$B \frac{rb}{Y} = \pi + \eta b$$

(103).

Hence, the differential equation for the debt ratio along the SSL which is the same as in the other two cases becomes:

$$b^{\sigma \pi_{SSL}} = \alpha \left( \pi + \left( b^{\sigma \pi_{SSL}} \right)^{SSL} \right) - \alpha (\delta + n) b^{\sigma \pi_{SSL}}$$

(104).

According to eqs (29), (59), and (64) with $\bar{\sigma} = 1$ we obtain:

$$\left( B^{(\pi_{SSL})} \right)^{SSL} = (1 - \alpha) \frac{sg}{s_{fr}} \omega \left( b^{\pi_{SSL}} \right)^m$$

(105).

---

8 The second area from below between the red and the blue line in Figure 8 (a) should be light red, not light blue.
and hence
\[ b^{\sigma z_{SSL}} = \alpha \left[ \pi + (1 - \alpha) s^g_{pr} \omega \left( b^{\sigma z_{SSL}} \right)^m \right] - \alpha(\delta + n)b^{\sigma z_{SSL}}; \]
\[ z = r, l, q, c; \quad m = 0.5, 1, 2, 3 \]  
(106)

with the slope
\[ \frac{\partial b^{\sigma z_{SSL}}}{\partial b^{\sigma z_{SSL}}} = \alpha \left[ (1 - \alpha) s^g_{pr} \omega \left( b^{\sigma z_{SSL}} \right)^{m-1} - (\delta + n) \right]; \quad z = r, l, q, c; \quad m = 0.5, 1, 2, 3 \]  
(107).

Depending on the exponent in the safety discount factor function and the values of the other parameters there are again up to three steady states. A steady state is stable if the phase curve of the debt ratio is negatively sloped, hence if
\[ \omega m \left( b^{\sigma z_{SSL}} \right)^{m-1} < \frac{\omega m \left( b^{\sigma z_{SSL}} \right)^{m-1} - (\delta + n)s^f_{pr}}{(1 - \alpha)s^g_{pr}} \]  
(108),
i.e. if the safety discount factor does not decrease too strongly with an increase in the debt ratio. If we compare the stability condition (108) with the stability condition (74) for the case with a fixed tax rate we see that the former is much stricter in the sense that the sensitivity of the safety discount factor with respect to an increase in the debt ratio has to be much smaller in absolute terms. Condition (108) can be rewritten:
\[ b^{\sigma z^*} < \left( m^{-1} \frac{\omega m \left( b^{\sigma z_{SSL}} \right)^{m-1} - (\delta + n)s^f_{pr}}{(1 - \alpha)s^g_{pr}} \right); \quad z = r, l, q, c; \quad m = 0.5, 1, 2, 3 \]  
(109).

and finally with eq. (105):
\[ b^{\sigma z^*} < \frac{(\delta + n)m}{m}; \quad z = r, l, q, c; \quad m = 0.5, 1, 2, 3 \]  
(110).

For the government to be solvent, the steady state rate of interest on government bonds has to exceed the steady state growth rate \( \delta + n \). Hence, only for \( m < 1 \) can steady state public debt be stable and the government’s solvency constraint be satisfied simultaneously (see Figure 9 (b)). But in this case the rate of interest falls with the debt ratio along the stable SSL, which contradicts empirical findings (see e.g. Laubach 2009 as well as Greenlaw et al 2013 and the literature referred to therein).

In the second steady state in Figure 9 (b) public debt \( b^{\sigma z_{2}} \) is long-term sustainable and efficient. But we see from Figure 9 (a) that the corresponding light grey area is rather small. An increase in the primary surplus ratio target would shift the phase curve of the debt ratio down in Figure 9 (b). Hence, the existence of at least one steady state with a positive public debt-to-GDP ratio as a precondition for public debt being sustainable would no longer be fulfilled even if a positive primary surplus guarantees the government’s solvency as the public debt ratio converges to zero. Yet, as households have a preference for holding a share of their wealth in government bonds this would not be optimal from the households’ point of view.
5. Concluding Remarks

The policy conclusion of this paper is straightforward. In order to make public debt long-term sustainable it is much more prudent to target a specific budget deficit ratio, not a specific primary surplus ratio. This is especially the case if the interest rate on government bonds rises with the debt ratio and if the debt ratio is already high. Targeting a surplus ratio target implies that public debt dynamics are determined to a large extent by market forces via the interest rate if the Central Bank does not intervene, whereas a realized budget deficit target directly determines public debt dynamics. Hence, the government can effectively control the debt dynamics with a budget deficit target, but not with a surplus ratio target.

The destabilization of steady states by primary surplus ratio targeting is especially problematic if the debt ratio is already high, as has been the case in the programme countries of the Euro area during the Eurozone debt crisis. This risks to set in motion an upward spiral of increasing debt ratios. Hence, the switch from budget deficit targeting to surplus ratio targeting occurred at the wrong time and in the wrong places.

Appendix

Steady state debt ratios with fixed tax rate:

\[
\frac{b^{\alpha \tau}}{s_{fr}^3} = \frac{1}{s_{fr}^2} s_{fr}^3 (1-\tau) \left( s_{fr}^f + s_{fr}^g \right) \left( 1 - \alpha \right) \omega
\]

\[
\frac{b^{\alpha \tau}}{s_{fr}^3} \left( (1-\tau) \left( s_{fr}^f + s_{fr}^g \right) \left( 1 - \alpha \right) \omega + \sqrt{(1-\tau)^2 \left( 4 s_{fr}^f (s_{fr}^f + s_{fr}^g) (s_{fr}^g)^3 (1-\alpha)^2 (1-\tau)^2 \right)} \right)
\]

\[
b^{\alpha \tau} = \frac{s_{fr}^f s_{fr}^g (1-\tau)}{s_{fr}^3 (s_{fr}^f + s_{fr}^g) (1-\tau) \left( 1 - \alpha \right) \omega}; \quad \frac{\partial b^{\alpha \tau}}{\partial \omega} > 0
\]
\[
\begin{align*}
b_{1}^{\sigma\alpha^*} &= \frac{s_{pr}^{f}(\delta + n) - \sqrt{s_{pr}^{f}\left[s_{pr}^{f}(\delta + n) + 4\left(s_{pr}^{g}\right)^{3}(1 - \alpha)(1 - \tau)^{2}\omega\right]}}{2\left(s_{pr}^{g}\right)^{2}(1 - \alpha)(1 - \tau)\omega} \\
&= \left\{ \frac{\sin \frac{\pi}{6}}{3} \arccos \left[ \frac{3\sqrt{3}s_{pr}^{f}}{2s_{pr}^{g}(1 - \alpha)\omega} \left(\frac{\left(s_{pr}^{g}\right)^{6}(1 - \alpha)^{3}(1 - \tau)^{3}\omega^{3}}{s_{pr}^{f}(\delta + n)^{3}}\right) \right] \right\} \tag{113}.
\end{align*}
\]

\[
\begin{align*}
b_{2}^{\sigma\alpha^*} &= -\frac{2}{\sqrt{3}} \frac{s_{pr}^{f}(\delta + n)}{2\left(s_{pr}^{g}\right)^{2}(1 - \alpha)(1 - \tau)\omega} \sin \frac{1}{3} \arccos \left[ \frac{\left(s_{pr}^{g}\right)^{6}(1 - \alpha)^{3}(1 - \tau)^{3}\omega^{3}}{2s_{pr}^{g}(1 - \alpha)\omega} \right] \\
&= \left\{ \frac{3\sqrt{3}s_{pr}^{f}}{2s_{pr}^{g}(1 - \alpha)\omega} \right\} \tag{114}.
\end{align*}
\]

**Steady state debt ratios with primary deficit ratio target:**

\[
\begin{align*}
b^{\sigma\pi\tau^*} &= \frac{s_{pr}^{g}(1 - \alpha)\omega}{2\left(s_{pr}^{g}\right)^{2}(\delta + n)^{2}} \left[ s_{pr}^{g}(1 - \alpha)\omega + \sqrt{s_{pr}^{g}\left(s_{pr}^{f}\right)^{2}(1 - \alpha)^{2}\omega^{2}4\pi\left(s_{pr}^{f}\right)^{2}(\delta + n)} \right] \\
&= \frac{-\pi s_{pr}^{f}}{s_{pr}^{g}(1 - \alpha)\omega - s_{pr}^{f}(\delta + n)}; \quad \frac{\partial b^{\sigma\pi\tau^*}}{\partial \omega} < 0 \\[10pt]
b^{\sigma\pi\pi^*} &= \frac{-2\pi s_{pr}^{f}}{\sqrt{s_{pr}^{f}\left(s_{pr}^{f}(\delta + n)^{2} - 4\pi s_{pr}^{g}(1 - \alpha)\omega\right)} - s_{pr}^{f}(\delta + n)}; \quad \frac{\partial b^{\sigma\pi\pi^*}}{\partial \omega} > 0 \\[10pt]
b_{2}^{\sigma\alpha^*} &= \frac{2}{\sqrt{3}} \frac{s_{pr}^{f}(\delta + n)\cos \frac{1}{3} \arccos \left[ \frac{3\sqrt{3}s_{pr}^{f}(1 - \alpha)^{3}\omega^{3}}{2s_{pr}^{g}(1 - \alpha)\omega} \right]}{s_{pr}^{g}(1 - \alpha)\omega} \tag{118}.
\end{align*}
\]

\(\text{pi refers to } \pi \text{ used in mathematics.}\)

\(\text{pi refers to } \pi \text{ used in mathematics.}\)
REFERENCES


