Can Real Public Debt Be Sustainable?
- A Contribution to the Theory of the Sustainability of Real Public Debt

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Abstract

The sustainability of government’s financial situation has three aspects: solvency, liquidity, and stability. The government is solvent if it satisfies its intertemporal budget constraint. In the steady state of a growing economy, the government is solvent if the public-debt-to-GDP ratio is constant and the real interest rate on government debt exceeds the growth rate of real GDP. The government is liquid if its instantaneous budget constraint is satisfied. In models of public debt dynamics with exogenously given primary-deficit-to-GDP ratio, growth and interest rate, this steady state debt-to-GDP ratio is unstable if the real interest rate on government debt exceeds the growth rate of real GDP. This may lead to an increase of the debt-to-GDP ratio and of the ratio of the government’s interest payments to GDP without bound, and hence to a violation of the government’s liquidity constraint. Hence, there seems to be a tension between sustainability concerning government’s solvency and sustainability concerning the steady state’s stability and hence government’s liquidity. This leads to the first research question of this paper: Is it true, that in a growing economy a constant debt-to-GDP ratio can only be stable if the government’s solvency constraint is violated? Or put differently: Is it true that real public debt cannot be sustainable? Obviously, one counterexample is enough to show that a proposition does not hold in general. Such a counterexample based on the Solow growth model is presented in the paper. Hence, sustainability of public debt is possible in the sense that both the solvency constraint and the liquidity constraint with a stable steady state real debt-to-GDP ratio are satisfied for a set of parameter constellations. Furthermore, again for an exogenous safety discount factor, an exogenous propensity to invest in government bonds, and an endogenous rate of interest, it is shown that the government can directly target a primary surplus-to-GDP ratio for the sake of long term debt sustainability, i.e. its solvency, without jeopardizing short term debt sustainability, i.e. its liquidity. But, in terms of time to adjust to the steady state, it is more efficient for the government to target a sufficiently low budget deficit-to-GDP ratio, thus implying a primary surplus in the steady state. Finally, if the rate of interest is fixed exogenously at a level larger than the growth rate, targeting a primary surplus ratio destabilizes the steady state and renders public debt unsustainable.

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1. Introduction

Public debt and its sustainability have become an acute problem since the beginning of the Great Recession in various countries, especially in the Euro Area. As, among others, Kindleberger and Aliber 2011 or Reinhart and Rogoff 2009 have pointed out, this is by no means a new phenomenon in economic history. Yet, there seem to exist different views on what are the conditions for the sustainability of public debt or more specifically the public debt-to-GDP ratio.

The most fundamental concept of the sustainability of public debt seems to be that the government remains solvent, i.e. that it satisfies its intertemporal budget constraint, named in the following: solvency constraint. As follows e.g. from Romer 2012 p. 586f, this is the case if the real interest rate on real government debt is always positive and larger than the rate of growth of real government debt. In a growing economy, this means that the government’s solvency constraint is satisfied if the public-debt-to-GDP ratio is constant and the real interest rate on government debt exceeds the growth rate of real GDP. Thus, with respect to solvency, a condition for sustainable public debt is a sufficiently high real rate of interest compared to the growth rate of real public debt which, for a constant debt-to-GDP ratio, just equals the growth rate of real GDP.

Yet, during the Euro crisis high rates of interest on government bonds, especially high spreads relative to the German Bund were considered a sign of fiscal weakness, as governments of countries like Greece, Ireland, Italy, Portugal, and Spain had to offer high yields in order to stay liquid by selling their bonds in financial markets. A government is liquid if its instantaneous budget constraint, named in the following liquidity constraint, is satisfied. In case of public deficits, satisfying the government’s liquidity constraint requires sufficient demand for newly issued government bonds if monetization of public debt is excluded. The government’s instantaneous budget constraint translates into a differential equation for the debt-to-GDP ratio from which an equilibrium debt-to-GDP ratio can be computed (see e.g. Gärtner 2009 p. 394f and de Grauwe 2014 annex; in the following: G2). According to Gärtner 2009 p. 395f, for a constant primary-deficit-to-GDP ratio, this (constant) steady state debt-to-GDP ratio is unstable if the real interest rate on government debt exceeds the growth rate of real GDP. In this case a shock that makes the actual debt-to-GDP ratio larger than its equilibrium value, leads to an ever increasing debt-to-GDP ratio and an ever increasing deficit-to-GDP ratio. In a closed economy without money (creation), this increasing deficit-to-GDP ratio cannot be financed out of private agents’ savings if their propensity to invest in government bonds is constant (and lower than unity). Hence, public debt becomes unsustainable, as the government becomes illiquid. Medium term fiscal sustainability concerning government’s liquidity in the sense of resilience to shocks requires the stability of the equilibrium debt-to-GDP ratio which, according to G2, requires the growth rate of real GDP to exceed the real rate of interest on government debt for a given primary-deficit-to-GDP ratio.

Hence, there seems to be a tension between sustainability concerning government’s solvency and sustainability concerning government’s liquidity which depends on the stability of the steady state debt-to-GDP ratio. This leads to the first research question of this paper: Is it true that in a growing economy a constant debt-to-GDP ratio can only be stable if the government’s solvency constraint is violated? An affirmative answer would imply that, by its very nature, public debt is unsustainable and, hence, has to be avoided. This would amount to a theoretical justification of a public debt brake.

Obviously, one counterexample is enough to show that a proposition does not hold in general. Such a counterexample is presented here. To this end the simple Solow growth model (Solow 1956) is generalized in order to include public capital and public debt as well as the government’s liquidity and solvency.

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1 This condition also plays an important role in the so-called fiscal theory of the price level (see Sims 2013 and the literature cited therein, especially Woodford 1995).
2 In the following time is assumed to be continuous.
3 Now and in the following, by stable we mean asymptotically stable in the sense that after a shock a system returns to its equilibrium (see e.g. Gandolfo 2010 pp 331ff). Contrary to e.g. European Central Bank 2011 p. 64, by stable we do not mean constant.
A growth model of the Solow type is chosen because its relevant steady state is asymptotically stable. Hence, the conditions under which the originally stable steady state becomes unstable can be studied. This research strategy obviously could not be followed if the starting point were a model whose steady state is either asymptotically unstable as a growth model of the Harrod-Domar type or a saddle point as an optimal growth model of the Ramsey type.

Whereas G2 treat the interest rate on public debt, the growth rate of GDP, and the primary-deficit-to-GDP ratio as exogenous variables, in the model presented below they will be endogenous. Here the stability of the (economically relevant) steady state and hence the equilibrium public debt ratio does not depend on whether the equilibrium growth rate of GDP exceeds the equilibrium rate of interest.

From an economic point of view the main difference between the model presented below and the model presented e.g. by G2 consists in the following: Whereas in G2’s model the debt dynamics are driven by the supply side (which might be called ‘Say’s Law of Public Finance’), namely the government’s need to finance a budget deficit, in the model presented below the debt dynamics are driven by the demand side, namely the demand of private households for government bonds, just as in the original Solow growth model private capital accumulation is driven by the savings of private households and hence the demand of private households for stock or bonds issued by private firms in order to finance private investments in real capital. It is assumed that the government’s financing needs and hence the supply of newly issued government bonds is so high that the quantity traded is determined by demand. This can be justified by reasons of political economy like elections that motivate governments to expand government expenditures in order to increase the probability of reelection. Furthermore, time and again comes the need to roll over government debt issued in the past. Additionally, we assume that the government tries to avoid default at all cost. Hence, the rate of interest on government bonds is also determined by demand.

In almost all countries in the world there has been public debt, at least during the last decades. In some countries sovereign debt crises were experienced, but luckily not in all. Hence in a “realistic” model for countries without sovereign debt crisis, that can describe and explain several stylized facts, there should exist steady states with a positive and stable debt-to-GDP ratio without violating the government’s solvency constraint. The latter requires the real rate of interest on public debt to exceed the growth rate of real GDP, which in turn requires a sufficiently large primary public budget surplus for a positive steady state debt-to-GDP ratio.

In the model presented below such steady states exist, as long as private households want to hold a part of their savings in government bonds as an asset with (under normal conditions) a positive rate of return and comparatively low risk. As in the original Solow model money is not taken into account. Hence there cannot be any liquidity preference in the sense of a demand for money. But the demand for government bonds can be viewed as stemming from a sort of liquidity or safety preference which induces private households to accept a lower rate of return, i.e. a safety discount for investments in government bonds compared to investments in bonds or equities issued by private firms. How strong this safety preference can become one could observe during the Euro crisis, when the nominal yield on bonds issued by the Federal Government of Germany sometimes became negative.

Presenting a model with a positive and stable steady state debt-to-GDP ratio, where the government is both solvent and liquid in the short and medium term due to its resilience to shocks and, hence, where its public debt is sustainable, does not mean to deny the possibility of unsustainable public debt. The model presented can thus form the base for identifying deeper reasons for the occurrence of fiscal crises. In the model private households’ propensity to invest in government bonds is assumed to be an

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4 As the Solow model does not include money, we deal with real debt in the sense of Sims 2013, p. 567f.
6 Domar 1944 is a seminal paper for the discussion of the sustainability of public debt in the context of economic growth. See for a treatise of public debt in an endogenous growth model Greiner and Fincke 2015.
7 First steps in this direction are undertaken in an accompanying paper by the author.
exogenous parameter just as the safety discount factor. These rather strong assumptions can be justified by three considerations: (i) they are in line with the original Solow growth model, where the saving rate is assumed to be exogenous; (ii) this is a paper on sovereign debt sustainability, not on sovereign debt crises including runs on government bonds; (iii) in another paper by the author both private households’ propensity to invest in government bonds and the safety discount factor will be endogenized.

In the model presented here the steady state and hence the steady state ratio of public debt to GDP turn out to be stable under rather general conditions, namely, whenever the sum of the rate of technical progress and the rate of change of labor supply, i.e. the equilibrium growth rate of real GDP, is positive and the partial production elasticity of labor in efficiency units lies within a range between zero and unity. Hence, stability does not depend on the sign of the difference between the growth rate and the rate of interest on government bonds.

The remainder of the paper is organized as follows: In section 2 the Solow growth model with public debt is presented. In section 3 we derive the system of differential equations, the steady state, and carry out various comparative dynamic analyses. In section 4 we will turn to the question whether it is really true that in a growing economy a constant debt-to-GDP ratio can only be stable and hence sustainable if the real rate of interest on government debt is lower than the growth rate of real GDP which in turn leads to a violation of the government’s solvency constraint. To answer this question we will first describe Romer’s approach (Romer 2012) and how it fits into the model presented here. Then we describe the approach by Gärtner 2009 and de Grauwe 2014 (G2), and finally we compare their approach with the one presented here.

In section 5 we consider the cases where the parliament writes into law rule-based tax rates which make sure that either a pre-established budget-deficit-to-GDP ratio or a pre-established primary-surplus-to-GDP ratio is continuously realized by setting the appropriate tax rate. In this way we want to investigate the impact of fiscal rules which have been established in parts of Europe: namely by the Treaty on Stability Coordination and Governance (see European Commission 2013) for member states of the Euro Area according to which the maximum budget deficit ratio is one percent, and even lower if the current public debt ratio exceeds sixty percent (see European Commission 2013, especially pp 94 ff. for details concerning their definition and computation). In the case of Greece the fiscal target does not refer to the public deficit-to-GDP ratio, but to the primary-surplus-to-GDP ratio instead (see European Commission 2012, p. 2 or European Commission 2015, pp 5ff.). It is shown that the government can directly target a primary-surplus-to-GDP ratio for the sake of long term debt sustainability without jeopardizing medium term debt sustainability, as long as the safety discount factor and the propensity to invest in government bonds are exogenous, and the rate of interest on government bonds is endogenous.

In section 6 we investigate what happens if the rate of interest on government bonds is no longer determined by domestic market forces, but exogenously, e.g. by international political agents like the European Commission, the European Central Bank and the International Monetary Fund in the case of the European Programme Countries (Greece, Ireland, Portugal, and Cyprus) or by international investors. Section 7 concludes.


Following Aschauer 1989 we take account of public capital $K_g$ in the production function

$$Y^S = \gamma A^\alpha L^\alpha K_p^\beta K_g^{1-\alpha-\beta}, \quad \gamma \geq 1$$

(1)
where $Y^S$ denotes aggregate supply of goods and services produced in the economy, $A$ technical efficiency, $L$ labor input, $K_{pr}$ private capital.\textsuperscript{8} We assume a Cobb-Douglas production function with constant returns to scale in order to be able to carry out numerical analyses. The main results of the analysis do not depend on the specific form of the production function as long as constant returns are preserved and the production function has a representation of the form $Y^S = F(AL,K_{pr},K_g)$, i.e. technical progress can be represented as purely labor-augmenting (see Acemoglu 2009, pp 58 ff.). For the sake of simplicity, the dependence of the variables on time is omitted in general. Total factor productivity grows at the rate of technical progress $\delta$, labor supply at the natural rate $n$. Labor supply is inelastic with respect to the real net wage. Capital and labor are fully employed:

$$A = A_0 e^{\delta t}; \quad \delta > 0$$
$$L = L_0 e^{nt}; \quad n > 0$$

Aggregate demand for goods and services ($Y^D$) consists of private ($C_{pr}$) and public consumption ($C_g$) as well as of net private ($I_{pr}$) and net public investment ($I_g$).

$$Y^D = C_{pr} + I_{pr} + C_g + I_g$$

As in the Solow model, the market for goods and services is in equilibrium:

$$Y^S = Y^D = Y$$

where $Y$ denotes net domestic product which equals net national income, as we consider a closed economy.\textsuperscript{9} Private consumption depends on available income of households ($Y - T + r_B B$)

$$C_{pr} = c_{pr} (Y - T + r_B B)$$

where $c_{pr}$ denotes households’ propensity to consume with respect to disposable income, $T$ net transfer payments from households to government (taxes plus social contributions from private households minus government’s transfer payments to private households), and $r_B$ the rate of interest on government bonds $B$. For households’ propensity to save obviously holds:

$$s_{pr} = 1 - c_{pr}$$

and hence for private savings $S_{pr}$

$$S_{pr} = s_{pr} (Y - T + r_B B)$$

In the original Solow model neither financial markets nor a banking sector are explicitly modelled. It is simply assumed that the financial market for firms’ (newly issued) bonds or equities is in equilibrium. Hence, (in the end) households’ savings are just invested by private firms. Now we have two financial markets: one for firms’ (newly issued) bonds or equities and one for (newly issued) government bonds. Accordingly, one part of savings $S^f_{pr}$ is invested in bonds/equities issued by private firms

\textsuperscript{8} This production function differs from the one e.g. in Barro 1990 and Minneia and Villieu 2012. These authors take ‘productive’ public expenditures into account, not public capital.

\textsuperscript{9} In the following, for brevity’s sake the term ‘net’ will be omitted in general. Just as in Solow’s original contribution to the theory of economic growth (Solow 1956) we do not treat depreciation explicitly in the model. But we will consider depreciation when we come to the Maastricht criteria concerning the debt-to-GDP and the public-deficit-to-GDP ratio.
\[ S^f_{pr} = s^f_{pr} (Y - T + r_B B) \] (9),

another in government bonds

\[ S^g_{pr} = s^g_{pr} (Y - T + r_B B) \] (10),

where

\[ s_{pr} = s^f_{pr} + s^g_{pr} \] (11).

Thus, in this as in Solow’s model bonds and equities are held directly by private households, whereas in reality government bonds are held to a large extent by intermediaries (see for recent data e.g. Andritzky 2012 and Broner et al. 2014). All public debt is domestic.

In the following, for brevity’s sake \( T \) will be called taxes, and the corresponding rate \( \tau \) tax rate. Furthermore, as in Solow’s original contribution we just stipulate consumption, saving, and investment functions without deriving them explicitly through a utility maximization exercise.\(^{11}\) In any case, implicitly we assume that households’ instantaneous utility depends on both consumption of goods and wealth which is invested in firm equities or bonds and government bonds according to the respective saving rates.

Taxes depend on the tax rate \( \tau \) and gross household income \( (Y + r_B B) \), which flows from private firms as labor and capital income and from government as interest payments on government bonds

\[ T = \tau (Y + r_B B) \] (12).

By postulating the saving functions (9) and (10), we assume that households’ preferences are such that they want to diversify their investments by investing their savings both in private equities/bonds and government bonds. Furthermore, we assume that private households just reinvest their financial means in firm and government bonds, whenever outstanding firm and government bonds become due. Alternatively we can assume that firms and governments issue perpetual bonds. Secondary markets for equities and government bonds are not explicitly modeled.

Savings that flow to private firms \( S^f_{pr} \) are used to finance private firms’ net investments \( I_{pr} \) in private real capital \( K_{pr} \)

\[ I_{pr} = K_{pr} = S^f_{pr} \] (13).\(^{12}\)

Savings that flow to government \( S^g_{pr} \) are used to buy new government bonds \( \dot{B}^D \) which finance the government’s budget deficit \( \dot{B} \):

\[ \dot{B} = \dot{B}^D = S^g_{pr} \] (14).

Here and in the following a dot above a variable denotes the first derivative with respect to time. Eq. (14) has the following economic implication: the government supplies additional government bonds that private households demand according to their disposable income and risk preferences. Government budget deficits respond to the portfolio choices of private households just as firms’ investments, i.e. firms’ deficits. Both are determined by the demand of private households for new assets. Government budget deficits are an equilibrium phenomenon just as private firms’ deficits financed by issuing stocks.

\(^{10}\) This assumption is in line with von Weizsäcker 2014.

\(^{11}\) See for a theoretical justification Akerlof’s Presidential Address at the Annual Meeting of the American Economic Association 2007 (Akerlof 2007, especially pp 13ff.).

\(^{12}\) Here and in the following we use the following abbreviations: \( \dot{x} = dx/dt \) and \( \ddot{x} = \dot{x}/x \).
or bonds. For the corresponding public-deficit-to-national-income ratio we get by simple division of eq. (14) by national income:

\[
\frac{\dot{B}}{Y} = \frac{s_{pr}^g}{Y} \tag{15}
\]

The government’s liquidity constraint is:

\[
T + \dot{B} = C_g + I_g + \tau_B B \tag{16}
\]

Excluding monetization of public debt, the government’s total revenue consists of taxes plus net increases in government debt \(\dot{B}\). This revenue is used to finance public expenditures on consumption \(C_g\) and investment \(I_g\) plus interest payments on outstanding government debt \(\tau_B B\).

The government’s primary deficit \(D\) is defined as follows:

\[
D = C_g + I_g - T \tag{17}
\]

and the corresponding ratio of the primary deficit to national income \(d\) as:

\[
d = \frac{D}{Y} = \frac{C_g + I_g - T}{Y} \tag{18}
\]

By using eqs (14) and (17) we obtain the following form of the government’s liquidity constraint:

\[
\dot{B} = D + \tau_B B \tag{19}
\]

With the public-debt-to-national-income ratio \(b\)

\[
b = \frac{B}{Y} \tag{20}
\]

we get from eqs (18) - (20):

\[
d = (\dot{B} - \tau_B) b \tag{21}
\]

where a hat over a variable indicates the percentage rate of change. For \(b > 0\) there is a primary deficit \((d > 0)\) if the rate of change of public debt exceeds the real rate of interest on public debt, and there is a primary surplus if the real rate of interest on public debt exceeds the rate of change of public debt.\(^{13}\)

From eqs (19), (20), (14), (12), and (10) we can also obtain another expression for the primary deficit ratio, namely:

\[
d = s_{pr}^g (1 - \tau) - \left(1 - s_{pr}^g (1 - \tau)\right) \tau_B b \tag{22}
\]

and hence with eq. (15):

\[
\frac{\dot{B}}{Y} = s_{pr}^g (1 - \tau)(1 + \tau_B b) \tag{23}
\]

Thus, the primary deficit ratio as well as the deficit ratio do not only depend on households’ propensity to invest in government bonds and the tax rate, but on the ratio of government’s interest payments to national income as well, as the latter influence households’ disposable income. From eq. (22) we can derive:

\(^{13}\) Contrary to Gärtner 2009 and de Grauwe 2014 the ratio of the primary deficit to national income is endogenous. As long as the liquidity constraint is satisfied, for a positive debt ratio, an exogenously given primary deficit or surplus directly determines whether the growth rate of public debt is larger, equal to or smaller than the rate of interest.
\[-d \leq r_B b \iff \tau \leq 1 \lor r_B b > 0 \lor s_{pr}^g > 0\]  
(24),

i.e. in case of a primary surplus \((d < 0)\) it is smaller than the government’s interest payments with a tax rate smaller than unity and a positive propensity to invest in government bonds. Otherwise, the government’s budget deficit would vanish or turn negative (see eq. (23)).

The budget deficit can be used to finance the government’s net investments in public real capital \(K_g\) or a part of them as well as to finance a part of public consumption or interest payments on outstanding government debt. We introduce the public-investment-to-budget-deficit ratio \(\lambda\) that is supposed to be set by the parliament when it decides on the budget:

\[\lambda = \frac{l_g}{B}\]  
(25).

In case of \(\lambda = 1\) the so-called Golden Rule of Public Finance is followed (see e.g. Bassetto and Sargent 2005 and 2006)\(^{14}\). For public capital accumulation we obtain:

\[\dot{K}_g = I_g = \lambda B; \quad \lambda > 0\]  
(26).

From eqs (16) and (26) we can deduce:

\[T = C_g + r_B B + (1 - 1/\lambda)l_g\]  
(27).

If the Golden Rule of Public Finance is followed, taxes just serve to finance public consumption and interest payments on public debt. From eq. (12) we get for the ratio of taxes to national income:

\[\frac{T}{Y} = \tau(1 + \frac{B}{Y})\]  
(28)

and hence with eq. (27) we get for the ratio of public consumption to national income:

\[\frac{C_g}{Y} = \tau - (1 - \tau)\frac{B}{Y} - (1 - 1/\lambda)\frac{l_g}{Y}\]  
(29).

From eqs (4) - (6), (8) - (10), (13), (20), (26) and (28) the following expression can be derived for the ratio of public consumption to national income:

\[\frac{C_g}{Y} = 1 - \left(1 - (1 - \lambda)s_{pr}^g\right)(1 - \tau)(1 + r_B b); \quad \frac{\partial C_g/Y}{\partial r_B b} < 0\]  
(30).\(^{15}\)

The ratio of public consumption to national income is an endogenous variable in this model which is determined by the government’s liquidity constraint. The higher the ratio of public investment to national income, the lower is the ratio of public consumption to national income. The liquidity constraint can only be satisfied if public consumption does not turn negative. This requires the ratio of public interest payments to national income to be smaller than or equal to:

\[\left(r_B b\right)_{\text{Max}}^C = \frac{1}{\left(1 - (1 - \lambda)s_{pr}^g\right)(1 - \tau)} - 1\]  
(31).

If \(\lambda = 1\) the maximum ratio of government’s interest payments to national income just depends on the tax rate:

\(^{14}\) See in this context also Buiter 2001, Blanchard and Giavazzi 2004 and Sachverständigenrat 2007 pp 3ff.

\(^{15}\) It should be noted that eqs (29) and (30) are not independent from each other.
Accordingly, the tax rate is a strong instrument for the government to stay liquid.

As in Solow’s growth model we assume perfect competition in the markets for goods and services, labor and private capital. The price level is assumed to be unity. As all incomes flow to private households, only they pay income tax. Hence, the rental price of private capital \( R \) equals the marginal productivity of private capital \( \left( Y_{K_{pr}} \right) \) and the wage rate equals the marginal productivity of labor \( \left( Y_L \right) \):

\[
R = Y_{K_{pr}} = \beta A^\alpha L^\alpha K_{pr}^{\beta-1} K_g^{1-\alpha-\beta} = \frac{\beta Y}{K_{pr}} \quad (33),
\]

\[
w = Y_{L} = \alpha A^\alpha L^\alpha K_{pr}^{\beta-1} K_g^{1-\alpha-\beta} = \frac{\alpha Y}{L} \quad (34).
\]

Furthermore, we assume that the use of public capital is free of charge. This leads to profits \( \Pi \) even with perfect competition:

\[
\Pi = (1 - \alpha - \beta) Y \quad (35).
\]

For the respective rate of profit \( r_{\Pi} \)

\[
r_{\Pi} = \frac{\Pi}{K_{pr}} \quad (36)
\]

we can compute from eq. (33):

\[
r_{\Pi} = (1 - \alpha - \beta) \frac{Y}{K_{pr}} \quad (37),
\]

and hence, for the overall rate of return on private capital \( r_K \)

\[
r_K = R + r_{\Pi} \quad (38).
\]

From eqs (33), (37), and (38) follows:

\[
r_K = \frac{(1 - \alpha) Y}{K_{pr}} = \frac{1 - \alpha}{\beta} Y_{K_{pr}} \quad (39).
\]

Even assuming perfect markets, we allow that the rate of return on private capital may exceed the one on government bonds due to risk and liquidity considerations of private households. This implies that households consider an investment in government bonds less risky and more liquid than an investment in private enterprises. Private households’ risk and liquidity considerations are taken into account by the exogenously given safety discount factor \( \sigma \) which is defined as the difference between the rate of return on private capital and the rate of interest on government bonds divided by the rate of return on private capital:

\[
\sigma = \frac{r_K - r_B}{r_K} \quad \sigma \leq 1 \quad (40)
\]

which leads to:

\[
r_B = (1 - \sigma) r_K \quad \sigma \leq 1 \quad (41),
\]

where \( (1 - \sigma) \) will be called ‘spread factor’. According to eqs (39) and (41) the rate of interest on government bonds is proportional to the marginal productivity of private capital. By assumption, the safety discount factor does not exceed unity. If the safety discount factor is close to one, public bonds are considered by private households as money which is held for speculative purposes in the sense of Keynes’
General Theory (Keynes 1951). Also from the government’s perspective, issuing government bonds becomes more similar to issuing money if the interest rate on government bonds approaches zero.

Finally, we define private and public capital-labor ratios in efficiency units:

\[ \kappa_{pr} = \frac{K_{pr}}{AL} \] (42),
\[ \kappa_{g} = \frac{K_{g}}{AL} \] (43),

as well as net domestic product per capita in efficiency units:\(^{16}\)

\[ y = \frac{Y}{AL} \] (44).

The production function can be rewritten by using eqs (44), (42) and (43):

\[ y = \frac{Y}{AL} = \gamma \kappa_{pr}^{\beta} \kappa_{g}^{1-\alpha-\beta} = y\left(\kappa_{pr}, \kappa_{g}\right) \] (45).

From eq. (26) we obtain

\[ K_{g} = \lambda B \] (46)

if \( \lambda \) remains unchanged over time, and hence according to eq. (20) for the debt ratio b:

\[ b = \frac{B/Y}{Y/AL} = \frac{K_{g}}{\lambda Y} = \lambda^{-1} \gamma^{-1} \kappa_{pr}^{-\beta} \kappa_{g}^{\alpha+\beta} ; \frac{\partial b}{\partial \kappa_{pr}} < 0 , \frac{\partial b}{\partial \kappa_{g}} > 0 \] (47).

For the interest rate on public debt we can derive from eqs (39) and (41)

\[ r_{B} = (1-\sigma)(1-\alpha) \frac{Y}{K_{pr}} = \gamma(1-\sigma)(1-\alpha) \kappa_{pr}^{-\beta} \kappa_{g}^{1-\alpha-\beta} ; \frac{\partial r_{B}}{\partial \kappa_{pr}} < 0 , \frac{\partial r_{B}}{\partial \kappa_{g}} > 0 \] (48),

and hence with eqs (48) and (47) for the ratio of government’s interest payments to national income:

\[ r_{B}b = \lambda^{-1}(1-\sigma)(1-\alpha) \frac{K_{g}}{\kappa_{pr}} ; \frac{\partial r_{B}b}{\partial \kappa_{pr}} < 0 , \frac{\partial r_{B}b}{\partial \kappa_{g}} > 0 \] (49),

and hence with eq. (12) for the ratio of households’ disposable income to national income:

\[ \frac{Y-T+r_{B}B}{Y} = (1-\tau)\left(1+ \lambda^{-1}(1-\sigma)(1-\alpha) \frac{K_{g}}{\kappa_{pr}}\right) \] (50).

From eqs (50), (9), and (10) we can deduce the ratios of households’ investment in private and public bonds to national income:

\[ \frac{S_{pr}}{Y} = s_{pr}(1-\tau)\left(1+ \lambda^{-1}(1-\sigma)(1-\alpha) \frac{K_{g}}{\kappa_{pr}}\right) ; \frac{\partial \left(\frac{S_{pr}}{Y}\right)}{\partial \kappa_{pr}} < 0 , \frac{\partial \left(\frac{S_{pr}}{Y}\right)}{\partial \kappa_{g}} > 0 \] (51),
\[ \frac{S_{g}}{Y} = s_{g}(1-\tau)\left(1+ \lambda^{-1}(1-\sigma)(1-\alpha) \frac{K_{g}}{\kappa_{pr}}\right) ; \frac{\partial \left(\frac{S_{g}}{Y}\right)}{\partial \kappa_{pr}} < 0 , \frac{\partial \left(\frac{S_{g}}{Y}\right)}{\partial \kappa_{g}} > 0 \] (52).

Moreover, from eqs (39) and (45) we derive for the rate of return on private capital:

\(^{16}\) For simplicity’s sake, in the following text we will often omit ‘in efficiency units’.
How the government’s interest payments ratio, the deficit ratio, the primary deficit ratio, and the debt ratio depend on the private and public capital-to-labor ratios, is shown in Figure 1:

We see that the deficit ratio does not vary much with the public-capital-to-labor ratio, whereas the government’s interest payments ratio, the primary deficit ratio, and the debt ratio do. From Figure 1 (a) we see that the primary deficit ratio is zero when the deficit ratio equals the government’s interest payments ratio.

Furthermore, we can compute the marginal productivity of public capital $Y_{Kg}$:

$$Y_{Kg} = \gamma(1-\alpha)k_{pr}^{\beta-1}k_{g}^{1-\alpha-\beta} = (1-\alpha-\beta)\frac{Y}{Kg}$$

Efficiency of public debt requires the shadow rate of return on the public deficit ($s_{rb}$) to be greater than or equal to the rate of interest on government bonds. The shadow rate of return on the public budget deficit is the weighted average of the marginal productivity of public capital ($Y_{Kg}$) and the marginal productivity of public consumption ($Y_{Cg}$) where the weights are determined by the public-investment-to-budget-deficit ratio $\lambda$:

$$s_{rb} = \lambda Y_{Kg} + (1-\lambda)Y_{Cg}$$

If $\lambda < 1$ the share $1-\lambda$ of the public deficit is used for financing public consumption whose marginal rate of productivity is assumed to be zero. Hence, we get for the shadow rate of return on the public budget deficit

$$s_{rb} = \lambda Y_{Kg}$$

and hence with eqs (46), (47), and (54):

$$s_{rb} = \frac{1-\alpha-\beta}{b}$$
According to eq. (57), the larger the debt ratio, the lower is the shadow rate of return on the public budget deficit.

3. Steady State and Comparative Dynamics

In this section we deal with growth dynamics and derive the system of differential equations for the private and the public capital-labor ratio and the steady state. We get from eq. (13) and eqs (42) - (44), and (50):

\[ \dot{\kappa}_{pr} = s_{pr}^f (1 - \tau) \gamma \left( \kappa_{pr}^{1 - \alpha - \beta} \kappa_{g}^{-1} - \lambda^{-1} (1 - \sigma) (1 - \alpha) \kappa_{pr}^{1 - \alpha - \beta} \kappa_{g}^{-1} \right) \]  

(58),

and from eq. (26) with eq. (10) and eqs (42) - (44), and (50):

\[ \dot{\kappa}_{g} = s_{pr}^g (1 - \tau) \gamma \left( \kappa_{pr}^{1 - \alpha - \beta} \kappa_{g}^{-1} + \lambda^{-1} (1 - \sigma) (1 - \alpha) \kappa_{pr}^{1 - \alpha - \beta} \kappa_{g}^{-1} \right) \]  

(59).

From eqs (42), (2) and (3) we can derive the percentage rate of change of the private-capital-labor ratio in efficiency units:

\[ \dot{\kappa}_{pr} = \dot{\kappa}_{g} - \delta - n \]  

(60),

and from eqs (43), (2) and (3) the percentage rate of change of the public-capital-labor ratio in efficiency units:

\[ \dot{\kappa}_{g} = \dot{\kappa}_{g} - \delta - n \]  

(61).

We obtain the system of differential equations by inserting eq. (58) into eq. (60):

\[ \dot{\kappa}_{pr} = s_{pr}^f (1 - \tau) \gamma \left( \kappa_{pr}^{1 - \alpha - \beta} \kappa_{g}^{-1} - \lambda^{-1} (1 - \sigma) (1 - \alpha) \kappa_{pr}^{1 - \alpha - \beta} \kappa_{g}^{-1} \right) - (\delta + n) \kappa_{pr} \]  

(62),

and eq. (59) into eq. (61):

\[ \dot{\kappa}_{g} = s_{pr}^g (1 - \tau) \gamma \left( \kappa_{pr}^{1 - \alpha - \beta} \kappa_{g}^{-1} + \lambda^{-1} (1 - \sigma) (1 - \alpha) \kappa_{pr}^{1 - \alpha - \beta} \kappa_{g}^{-1} \right) - (\delta + n) \kappa_{g} \]  

(63).

The steady state \((\kappa_{pr}^*, \kappa_{g}^*)\) is derived by setting both eqs (62) and (63) equal to zero. By dividing the two resulting equations, we can see that the following relationship holds in the steady state:

\[ \kappa_{pr} = \frac{s_{pr}^f}{s_{pr}^g} \kappa_{g} \]  

(64).

Eq. (64) will be called steady state locus (SSL). On the SSL the proportion of private to public capital is fixed and given by the ratio of the respective propensities to invest and the public-investment-to-budget-deficit ratio. By inserting eq. (64) into eqs (58) and (59) it can be shown that the rates of change of public and private capital and hence of the public and the private capital-to-labor ratios are equal on the steady state locus. Hence, (i) a trajectory in a \(\kappa_{pr} - \kappa_{g}\) phase diagram that starts on the SSL remains on the SSL; (ii) the SSL cannot be crossed by a trajectory.\(^ {17}\) We define the ratio of private to public capital:

\[ \kappa_{prg} = \frac{\kappa_{pr}}{\kappa_{g}} \]  

(65).

\(^ {17}\) The steady state locus is not altered if a budget deficit-to-national-income ratio or a public surplus-to-national-income ratio is targeted.
whose SSL value is:

\[
\left( \kappa_{pr}^{g} \right)_{\text{SSL}} = \frac{s_{pr}^f}{\lambda s_{pr}^g} \tag{66}
\]

With eqs (62) and (63) we can derive the corresponding differential equation

\[
\dot{\kappa}_{pr}^{g} = (\dot{\kappa}_{pr}^{n} - \dot{\kappa}_{g}^{n}) \kappa_{pr}^{g} = (1 - \tau) \gamma \left[ 1 + \lambda^{-1} (1 - \sigma) (1 - \alpha) (\kappa_{g}^{pr})^{-1} \right] \kappa_{g}^{-\alpha} (\kappa_{g}^{pr})^{\beta} \left( s_{pr}^f - \lambda s_{pr}^g \kappa_{g}^{pr} \right) \tag{67}
\]

Obviously, we get:

\[
\kappa_{pr}^{g} = 0 \quad \forall \quad \kappa_{g}^{pr} = \left( \kappa_{g}^{pr} \right)_{\text{SSL}} \tag{68}
\]

Hence, on the SSL the ratio of private to public capital is constant. For a given public-capital-to-labor ratio, we can derive the slope of the differential equation on the SSL:

\[
\frac{\partial \kappa_{pr}^{g}}{\partial \kappa_{pr}^{g}} = -\lambda s_{pr}^g (1 - \tau) \gamma \left[ 1 + (1 - \sigma)(1 - \alpha) \left( \frac{s_{pr}^f}{s_{pr}^g} \right)^{-1} \right] \left( \frac{s_{pr}^f}{\lambda s_{pr}^g} \right)^{\beta} \kappa_{g}^{-\alpha} < 0 \tag{69}
\]

which is negative. Accordingly, the SSL is at least locally stable for any given positive public-capital-to-labor ratio. But from Figure 2 we see that the SSL is globally stable.

Furthermore, we define the ratio of the propensities to save/invest:

\[
s_{pr}^{fg} = \frac{s_{pr}^f}{s_{pr}^g} \tag{70}
\]

By inserting eq. (64) into eqs (51) and (52) we can derive the steady state values of the ratios of households’ investment in private and public bonds to national income:
\[ s_{pr}^f = s_{pr}^f (1 - \tau) \left( 1 + (1 - \alpha)(1 - \alpha) \left( s_{pr}^g \right)^{-1} \right); \]
\[ \frac{\partial s_{pr}^f}{\partial \tau} < 0, \quad \frac{\partial s_{pr}^f}{\partial \sigma} < 0, \quad \frac{\partial s_{pr}^f}{\partial \alpha} < 0, \quad \frac{\partial s_{pr}^f}{\partial s_{pr}} > 0, \quad \frac{\partial s_{pr}^f}{\partial s_{pr}^g} > 0 \]

and

\[ s_{pr}^g = s_{pr}^g (1 - \tau) \left( 1 + (1 - \sigma)(1 - \alpha) \left( s_{pr}^g \right)^{-1} \right); \]
\[ \frac{\partial s_{pr}^g}{\partial \tau} < 0, \quad \frac{\partial s_{pr}^g}{\partial \sigma} < 0, \quad \frac{\partial s_{pr}^g}{\partial \alpha} < 0, \quad \frac{\partial s_{pr}^g}{\partial s_{pr}} < 0, \quad \frac{\partial s_{pr}^g}{\partial s_{pr}^g} > 0 \]

Finally, in the steady state, we can rewrite eqs (62) and (63) in the following way:

\[ \dot{s}_{pr}^f Y^* (\kappa_{pr}^*, \kappa_g^*) = (\delta + n) \kappa_{pr}^* \]  
(73),

and

\[ \lambda \dot{s}_{pr}^g Y^* (\kappa_{pr}^*, \kappa_g^*) = (\delta + n) \kappa_g^* \]  
(74).

Eqs (73) and (74) are the equivalents to the well-known equilibrium condition in the Solow model: \( s_{pr}^f Y^* (\kappa_{pr}^*) = (\delta + n) \kappa_{pr}^* \). Households' propensity to save and invest in private capital with respect to national income \( \dot{s}_{pr}^f \) decreases with the tax rate and the safety discount factor, because in both cases households' disposable income shrinks. The same holds for the households' propensity to save and invest in government bonds with respect to national income \( \dot{s}_{pr}^g \) which, according to eq. (15), equals the steady state public-deficit-to-national-income ratio:\(^{18}\)

\[ \left( \frac{B}{Y} \right)^* = s_{pr}^g = s_{pr}^g (1 - \tau) \left( 1 + (1 - \sigma)(1 - \alpha) \left( s_{pr}^g \right)^{-1} \right) \]  
(75).

From eq. (75) we see that the steady state public deficit ratio is constant and independent from the growth rate, just as in the original Solow growth model the ratio of the change of capital to national income is equal to the savings rate which is constant (\( \dot{K}_{pr} / \dot{Y} = s_{pr}^f \)).\(^{19}\)

We obtain the steady state values of public and private capital per capita by inserting eq. (64) into eq. (62):\(^{20}\)

\(^{18}\) Below it will be shown that the steady state public-deficit-to-national-income ratio prevails not just in the steady state, but along the steady state locus in general.

\(^{19}\) With a more general production function than the Cobb-Douglas production function the sum of the production elasticities of private and public capital \( 1 - \alpha \) and hence the public deficit ratio would not be exogenously given.

\(^{20}\) As in the original Solow model there is a second steady state, namely \( \kappa_g = \kappa_{pr} = 0 \). This steady state is an unstable node.
The steady state values of the private and public capital-labor ratios are positive if all the terms in eqs (76) and (77) are positive which implies the tax rate and production elasticity of labor to be smaller than unity. Figure 3 shows how the steady state values of the private and public capital-labor ratios can be determined graphically in the three-dimensional Solow Diagram in the intersection point of the two differential equations and the $\kappa_{pr} = \kappa_g = 0$-plain. Through the steady state runs the steady state locus, too.

![Three-dimensional Solow Diagram](image1)

![Two-dimensional Solow Diagram](image2)

Figure 3: Steady state values of the private, public and aggregate capital-to-labor ratios

\[
(s_g = 0.02; s_{pr} = 0.1; \gamma = 1; \alpha = 0.7; \beta = 0.2; \tau = 0.32; n = 0; \delta = 0.02; \sigma = 0.6; \lambda = 1)^{21}
\]

As the steady state locus is globally stable, it makes sense to consider the case where the initial values of private and public capital lie on the SSL. If we insert the steady state locus (64) as well as the equations (70), and (72) into eq. (63) we obtain the differential equation for the public-capital-to-labor ratio:

\[
\frac{\partial \kappa_{pr}^*}{\partial (\delta + n)} < 0, \frac{\partial \kappa_{pr}^*}{\partial \tau} < 0, \frac{\partial \kappa_{pr}^*}{\partial \sigma} < 0, \frac{\partial \kappa_{pr}^*}{\partial \alpha} < 0, \frac{\partial \kappa_{pr}^*}{\partial s_{pr}} > 0, \frac{\partial \kappa_{g}^*}{\partial s_{pr}} > 0, -\frac{\partial \kappa_{g}^*}{\partial \lambda} > 0
\]

and finally, again using eq. (64):

\[
\kappa_{pr}^* = \left[\gamma \lambda^{1-\beta} \left(s_{pr}^g\right)^\beta \left(s_{pr}^g\right)\right]^{\frac{1}{\alpha}}
\]

\[
\frac{\partial \kappa_{pr}^*}{\partial (\delta + n)} < 0, \frac{\partial \kappa_{pr}^*}{\partial \tau} < 0, \frac{\partial \kappa_{pr}^*}{\partial \sigma} < 0, \frac{\partial \kappa_{pr}^*}{\partial \alpha} < 0, \frac{\partial \kappa_{pr}^*}{\partial s_{pr}} > 0, \frac{\partial \kappa_{g}^*}{\partial s_{pr}} > 0, -\frac{\partial \kappa_{g}^*}{\partial \lambda} > 0
\]

\[
\kappa_{g}^* = \left[\gamma \lambda^{1-\beta} \left(s_{pr}^g\right)^\beta \left(s_{pr}^g\right)\right]^{\frac{1}{\alpha}}
\]

(76),

(77).

21 The parameter values in the various figures are based on rough estimates for Germany.
The structure of the differential equation for the public-capital-to-labor ratio is the same as the structure of the differential equation for the capital-to-labor ratio in the original Solow paper (Solow 1956). On the SSL the composition of the aggregate capital stock \( \kappa \) which consists of private and public capital is fixed. For aggregating private and public capital we use:

\[
\kappa = \kappa^{pr}(1-\alpha)\kappa^g(1-\alpha)/(1-\alpha)
\]  

(80).

With a starting point on the SSL we get with eq. (64):

\[
\kappa^{SSL} = \lambda^{-1}\frac{fg}{sp}k^{SSL}
\]  

(81).

and hence

\[
\kappa^{SSL} = \frac{\bar{s}_pg}{\bar{s}_pr} \gamma \left( \frac{\kappa^{SSL}}{\delta+n} \right)^{1-\alpha} - (\delta+n)\kappa^{SSL}
\]  

(82),

where

\[
\frac{\bar{s}_pg}{\bar{s}_pr} = \lambda^{-1}\frac{fg}{sp}\left( \frac{fg}{sp} \right)^{\beta/(1-\alpha)} \frac{\kappa^g}{\kappa^{pr}}
\]  

(83)

which equals the ratio of aggregate investment to national income on the steady state locus. Eq. (82) is the differential equation for the aggregate-capital-to-labor ratio, which is shown in Figure 3 (b). By setting eq. (83) equal to zero we derive the steady state value of the aggregate capital-to-labor ratio:

\[
\kappa^* = \left( \frac{\bar{s}_pg}{\bar{s}_pr} \right)^{1/\alpha} \gamma \lambda^{-1}\frac{fg}{sp}\left( \frac{fg}{sp} \right)^{\beta/(1-\alpha)} \frac{\kappa^g}{\kappa^{pr}} \left( \frac{\kappa^{SSL}}{\delta+n} \right)^{1/\alpha}
\]  

(84).

By inserting eqs (76) and (77) into eq. (45) and taking account of eq. (72) the steady state value of national income per capita follows:

\[
y^* = \gamma \lambda^{-1}\frac{fg}{sp}\left( \frac{fg}{sp} \right)^{\beta/(1-\alpha)} \frac{\kappa^g}{\kappa^{pr}} \left( \frac{\kappa^{SSL}}{\delta+n} \right)^{1/\alpha}
\]  

(85).

In analogy to the simple Solow model (Solow 1956), according to eq. (85), the steady state national income per capita increases with an increase in the ratio of aggregate investment to national income and a decrease in the equilibrium growth rate \( \delta+n \). Accordingly, steady state per capita income increases with a proportional increase in the propensities to save, an increase in the public-investment-to-budget-deficit ratio \( \lambda \), and with decreases in the tax rate \( \tau \) and the safety discount factor \( \sigma \).

Along the SSL the debt ratio changes with the aggregate-capital-to-labor ratio, as we obtain from eqs (47), (64), and (81):

\[
b^{SSL} = \gamma^{-1}\lambda^{-1}(1-\alpha-\beta)/(1-\alpha)\left( \frac{fg}{sp} \right)^{\beta/(1-\alpha)} \left( \kappa^{SSL} \right)^{\alpha}
\]  

(86).
The larger the aggregate-capital-to-labor ratio, the larger is the debt ratio. Obviously, in the steady state the growth rates of national income, private and public capital as well as public debt coincide with the growth rate of labor supply in efficiency units:

$$\dot{Y}^* = \dot{K}_{pr}^* = \dot{K}_{g}^* = \dot{B}^* = \delta + n$$

(87).

By inserting eqs (76) and (77) into eq. (47), and using eq. (72), we get for the steady state value of the ratio of public debt to national income:

$$b^* = \frac{s_{pr}^g (1-\tau)}{\delta + n} \left[ 1 + (1-\sigma)(1-\alpha) \left( \frac{s_{pr}^g}{s_{pr}^g} \right)^{-1} \right] = \frac{s_{pr}^g}{\delta + n};$$

(88).

Eq. (88) shows the well-known result that the steady state value of the debt ratio is equal to the steady state public deficit ratio divided by the steady state growth rate of national income. Under the additional assumptions $s_{pr}^g > 0$ and $\delta + n > 0$ the steady state debt ratio $b^*$ is also positive. According to eq. (88) the steady state debt ratio is the larger, the larger the households’ propensity to save and invest in government bonds ($s_{pr}^g$) and the larger the sum of the partial production elasticities of private and public capital $(1-\alpha)$. The steady state debt ratio is the smaller, the larger the equilibrium growth rate $(\delta + n)$, the tax rate $(\tau)$, the safety discount factor $(\sigma)$, and the households’ propensity to save and invest in shares of private firms ($s_{pr}^g$). What may be surprising at first sight is the fact that the steady state debt ratio does not depend on the public-investment-to-budget-deficit ratio $\lambda$. But this fact is in line with the government budget deficit being determined by private households’ demand for government bonds as an asset for their savings. Government can influence the steady state public deficit ratio and hence the steady state debt ratio only by varying the tax rate, thus influencing disposable income and hence households’ demand for newly issued government bonds.

Figure 4: Steady state public deficit ratio and the Maastricht criterion

( $s_{pr}^g = 0.1; \gamma = 1; \alpha = 0.7; \beta = 0.2; n = 0; \delta = 0.02; \sigma = 0.6; \lambda = 1$)

Figure 4 shows how the steady state public deficit ratio varies mainly with households’ propensity to invest in government bonds. The respective slope is close to unity. It is the larger the smaller is the tax

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22 The signs of the impacts of variations of the growth rate and the tax rate on the steady state public debt ratio are the same as in the simple public debt dynamics model of Gärtner 2009 in the case of a positive stable equilibrium public debt ratio.
rate. The Maastricht criterion of 3% refers to the debt-to-GDP ratio. The corresponding criterion for the public-deficit-to-NDP ratio would be about 15 – 20% higher because of $\frac{B}{Y_Y}$ referring to GDP and $(B/Y)^{9/g}$ the public-deficit-to-GDP ratio. The same reasoning applies to the debt ratio. Yet, for simplicity’s sake, in the figures we neglect the distinction between public-deficit-to-NDP ratio and public-deficit-to-GDP ratio.

By multiplying eqs (85) and (88) or by dividing eq. (76) by $\lambda$, we can derive the steady state value of public debt per capita:

$$y^* = \gamma^{1/\alpha} (1-\alpha - \beta) \alpha (\delta + n)^{-1/\alpha} \left( \frac{s_{pr}^g}{s_{pr}} \right)^{\gamma/\alpha} \left( \frac{g_{pr}^r}{g_{pr}} \right)^{1/\alpha}$$

$$\frac{\partial (y^*)}{\partial (\delta + n)} < 0, \frac{\partial (y^*)}{\partial \tau} < 0, \frac{\partial (y^*)}{\partial \sigma} < 0, \frac{\partial (y^*)}{\partial \alpha} < 0, \frac{\partial (y^*)}{\partial s_{pr}^f} > 0, \frac{\partial (y^*)}{\partial s_{pr}^g} > 0, \frac{\partial (y^*)}{\partial \lambda} > 0 \quad (89).$$

Even if, as pointed out above, the steady state value of the ratio of public debt to national income remains unaffected by changes in $\lambda$, this does not hold for the steady state value of public debt per capita. This increases with $\lambda$ as does per capita income.

By inserting eq. (64) into eq. (49) we derive along the SSL and for the steady state value of the ratio of public interest payments to national income:

$$f_{BSSL} = \frac{s_{pr}^g (1-\sigma) (1-\alpha)}{s_{pr}}.$$

Hence, along the SSL the ratio of public interest payments to national income is constant and equal to the steady state value. According to eqs (23) and (75), the same holds for the budget deficit ratio. For the steady state value of the ratio of public interest payments to national income to be smaller than unity it is sufficient that households invest at least as much in private firms as in government bonds. According to eq. (90) there is a negative relationship between the steady state values of the interest rate on government bonds and the public debt ratio along the SSL:

$$f_{BSSL} = \frac{s_{pr}^g (1-\sigma) (1-\alpha)}{s_{pr}^f} \left( b_{SSL} \right)^{-1}; \frac{\partial f_{BSSL}}{\partial \sigma} < 0 \quad (91).$$

The lower the public debt ratio, the higher is the rate of interest on government bonds, given the safety discount factor, labor’s partial production elasticity, and the ratio of saving rates. Remarkably, government cannot influence the ratio of public interest payments to national income by changing the tax rate. In fact, in the presented model the steady state value of the ratio of public interest payments to national income is determined by preferences and risk considerations of private households and by technology, not by government. 23 In the special case of identical propensities to invest ($s_{pr}^g = s_{pr}^f$) and a safety dis-

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23 This result depends on the production function being of the Cobb-Douglas-type. By differentiating eq. (90) with respect to the tax rate we get $d(f_{BSSL})/d\tau = df_{BSSL}/d\tau f_{BSSL} + f_{BSSL} df_{BSSL}/d\tau = 0$ and hence the elasticity of the rate of interest on government bonds with respect to the debt ratio in the steady state equals minus unity: $e_{f_{BSSL}} = df_{BSSL}/db^* \left( b^*/f_{BSSL} \right) = -1$. 

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count factor equal to zero, the ratio of public interest payments to national income just equals private capital’s share in national income.\(^{24}\)

If we furthermore divide eq. (90) by eq. (88) we finally obtain the steady state value of the interest rate on public debt:

$$f_B^* = \frac{(\delta + n)(1-\alpha)(1-\alpha)}{(1-\tau)\left(s_{pr}^f + s_{pr}^g(1-\sigma)(1-\alpha)\right)} = \frac{(\delta + n)(1-\alpha)}{s_{pr}^f s_{pr}^g};$$

(92).

If we further divide eq. (92) by eq. (88) we obtain

$$\frac{\partial f_B^*}{\partial (\delta + n)} > 0, \frac{\partial f_B^*}{\partial \tau} > 0, \frac{\partial f_B^*}{\partial \alpha} < 0, \frac{\partial f_B^*}{\partial s_{pr}^f} < 0, \frac{\partial f_B^*}{\partial s_{pr}^g} < 0, \frac{\partial f_B^*}{\partial \lambda} = 0.$$

Eq. (92) is the analogous expression of the steady state value of the real rental price of (private) capital in the original Solow growth model, namely \(r_K = (\delta + n)(1-\alpha)/s_{pr}^f.\)\(^{25}\) According to eq. (92) the steady state rate of interest on government bonds decreases with an increase in the propensities to save and the safety discount factor \(\sigma\) as well as with a decrease in the equilibrium growth rate \((\delta + n)\) and the tax rate \(\tau\). Neither the steady state value of the ratio of public interest payments to national income nor the steady state rate of interest on government bonds depend on the ratio of public investment to the budget deficit \(\lambda\).

From eqs (33), (39), (41), (54), (57), and (64) we get:

$$f_B^{SSL} = (1-\sigma)r_K^{SSL} = (1-\sigma)\frac{1-\alpha}{\beta} Y_{K_{pr}}^{SSL} = (1-\alpha)\frac{1-\alpha}{1-\beta - \beta} \frac{s_{pr}^g}{s_{pr}^f} Y_{B_{g}}^{SSL};$$

(93).

According to eqs (93), which also hold in the steady state, the ratios of the rate of interest on government bonds to the rate of return on private capital, to the marginal productivity of private capital, and to the marginal productivity of public capital are constant along the SSL. Hence, along the SSL, the rate of return on private capital as well as the marginal productivity of private and public capital, and the shadow rate of return on the public budget deficit are negatively correlated with the debt ratio, too (see eq. (91)).

Whether the rate of interest on government bonds is smaller than the shadow rate of return on the public budget deficit largely depends on the ratio of the propensities to invest in public and private bonds/equities \(s_{pr}^g/s_{pr}^f\). In the following we assume

$$\sigma \geq \sigma_{Min}^\text{eff} = 1 - \frac{(1-\alpha-\beta)}{(1-\alpha)} s_{pr}^f s_{pr}^g \iff f_B^* \leq s_{B_{g}}^* = \lambda Y_{K_{g}}^* = \frac{1-\alpha-\beta}{b^*};$$

(94),\(^{26}\)

a case which is shown in Figure 5 and Figure 6. Furthermore, from Figure 5 (a) we can see that the steady state growth rate exceeds the steady state rate of interest only for rather high values of the safe-
ty discount factor. By using eqs (22), (23) and (90) both the value of the primary deficit ratio \(d\) along
the SSL as well as the budget deficit ratio can be computed as:

\[
d_{\text{SSL}} = d^* = s_{pr}^g (1 - \tau) - (1 - \sigma)(1 - \alpha) \left( 1 - s_{pr}^g (1 - \tau) \right) \left( s_{pr}^g \right)^{-1};
\]

\[
\frac{\partial d^*}{\partial s_{pr}^f} > 0, \quad \frac{\partial d^*}{\partial s_{pr}^g} > 0, \quad \frac{\partial d^*}{\partial \tau} < 0, \quad \frac{\partial d^*}{\partial \sigma} > 0, \quad \frac{\partial d^*}{\partial \alpha} > 0, \quad \frac{\partial d^*}{\partial \lambda} = 0, \quad \frac{\partial d^*}{\partial (\delta + n)} = 0
\]

and

\[
\frac{\partial (\dot{B}/Y)^*}{\partial s_{pr}^f} < 0, \quad \frac{\partial (\dot{B}/Y)^*}{\partial s_{pr}^g} > 0, \quad \frac{\partial (\dot{B}/Y)^*}{\partial \tau} < 0, \quad \frac{\partial (\dot{B}/Y)^*}{\partial \sigma} < 0, \quad \frac{\partial (\dot{B}/Y)^*}{\partial \alpha} < 0, \quad \frac{\partial (\dot{B}/Y)^*}{\partial \lambda} = 0, \quad \frac{\partial (\dot{B}/Y)^*}{\partial (\delta + n)} = 0
\]

(a) Steady state values of the shadow rate of return on the public deficit, of the rate of the interest, and of the growth rate of national income

(b) Steady state values of the primary deficit ratio and of the difference between the growth rate of real national income and the real interest rate on public debt

![Figure 5: Steady states: rate of interest on government bonds, primary deficit ratio and growth rate of real national income](image)

The primary deficit ratio along the SSL, which can be positive or negative or zero, is constant and equal to its steady state level. Hence, it does not vary with the debt ratio along the SSL which holds for the budget deficit ratio, too. According to eq. (91) the rate of interest along the SSL falls with the safety discount factor, whereas the primary deficit ratio rises (see eq. (95)). From eq. (21) we directly get the following relationship between the steady state values of the primary deficit ratio, the debt ratio, and the difference between the growth rate of public debt and the real rate of interest:

\[
d^* = \left( \frac{\dot{B}}{Y} \right)^* - f^*_B = (\dot{b}^* - f^*_B) b^*
\]

Taking into account eq. (87), we can derive for \(b^* > 0\):

\[
d^* \geq 0 \iff \dot{Y}^* = \dot{B}^* \geq f^*_B
\]

Hence, for \(b^* > 0\), in the steady state there is a primary deficit if the growth rate of real national income exceeds the real rate of interest on public debt, and there is a primary surplus if the growth rate of real national income is smaller than the real rate of interest on public debt. But according to eq. (95) this does not imply that a change in the steady state growth rate \((\delta + n)\) affects the steady state primary
deficit ratio, as an increase in the difference between the steady state growth rate and the steady state rate of interest is just compensated by a decrease in the steady state debt ratio (see Figure 6).  

27 If the growth rate tends to zero the steady state interest rate tends to zero as well, whereas the steady state debt ratio tends to infinity, keeping the ratio of public interest payments to national income constant. This is an implication of the production function being of the Cobb-Douglas type.
\[ \left( \frac{C_g}{Y} \right)^{\text{SSL}} = \left( \frac{C_g}{Y} \right)_{1 - (1 - \lambda) \frac{g_{pr}}{s_{pr}}} = 1 - \left( 1 - \left( 1 - \left( 1 - \sigma \right) \right) s_{pr} \right) Y \frac{g_{pr}}{s_{pr}} \left( 1 - \left( 1 - \left( 1 - \alpha \right) \right) \right) \frac{\partial \left( C_g / Y \right)^{\ast}}{\partial \left( \delta + n \right)} = 0, \]

\[ \frac{\partial \left( C_g / Y \right)^{\ast}}{\partial \tau} > 0, \frac{\partial \left( C_g / Y \right)^{\ast}}{\partial \sigma} > 0, \frac{\partial \left( C_g / Y \right)^{\ast}}{\partial \alpha} > 0, \frac{\partial \left( C_g / Y \right)^{\ast}}{\partial s_{pr}} > 0, \frac{\partial \left( C_g / Y \right)^{\ast}}{\partial g_{pr}} < 0, \frac{\partial \left( C_g / Y \right)^{\ast}}{\partial \lambda} < 0 \]

(101).

Hence, the steady state ratio of public consumption to national income is a positively sloped linear function of the tax rate. It can become negative for a sufficiently low tax rate and approaches unity if the tax rate approaches unity. There exists no Laffer curve, as factor supplies are inelastic with respect to factor prices.

If a debt brake is introduced as in e.g. Germany (see Feld 2010), where the debt-brake-public-deficit-to-GDP ratio is restricted to 0.35 % in the constitution:

\[ \left( \frac{B}{Y} \right)^{DB}_{GER} = \left( \frac{s_{pr}}{Y} \right)^{DB}_{GER} = \frac{0.035 V_{GER}^{Y}}{Y_{GER}}, \]

(102),

with DB referring to debt brake and GER to Germany, the corresponding debt-brake-tax rate \( \tau^{DB} \) can be computed from eq. (72):

\[ \tau^{DB} = 1 - \frac{\left( \frac{s_{pr}}{s_{pr}} \right)^{DB}}{1 + (1 - \sigma) (1 - \alpha) \left( \frac{s_{pr}}{s_{pr}} \right)^{-1}} \frac{\partial \tau^{DB}}{\partial \sigma} < 0, \frac{\partial \tau^{DB}}{\partial \alpha} < 0, \frac{\partial \tau^{DB}}{\partial s_{pr}} < 0, \frac{\partial \tau^{DB}}{\partial g_{pr}} > 0, \frac{\partial \tau^{DB}}{\partial \lambda} = 0 \]

(103).

From eq. (103) we can see that the debt brake tax rate increases with households’ propensity to invest in government bonds. Thus, in order to satisfy the debt brake, the government has to increase the tax rate when private households increase their propensity to invest in government bonds, thus directly acting against private households’ preferences. The debt brake tax rate decreases with an increase of the safety discount factor, because an increase in the safety discount factor reduces households’ interest income from government bonds and this lowers households’ propensity to save and invest in government bonds with respect to national income.

By inserting the debt-brake-public-deficit ratio into eq. (88) we can compute the corresponding steady state debt-brake-public-debt ratio:

\[ b^{DB} = \frac{\left( \frac{s_{pr}}{s_{pr}} \right)^{DB}}{\delta + n} \]

(104).\(^{28}\)

According to eq. (104) a decrease of the public deficit ratio due to the introduction of a debt brake does not only reduce the steady state public debt ratio, but according to eq. (85) also per capita income unless government increases the public-investment-to-budget-deficit ratio sufficiently in order to counterbalance the adverse impact. By differentiating eqs (85) and (72) we can compute the percentage change of the public-investment-to-budget-deficit ratio \( \lambda \), which is necessary in order to keep national income per capita unchanged if the tax rate is increased in order to reduce the public deficit ratio:

\(^{28}\) If we assume an average growth rate of nominal GDP of about 3 % for Germany the resulting long run debt brake public debt ratio is about 10 %.
4. Sustainability of Public Debt Dynamics

We now turn to the question posed in the introduction, namely: Is it really true that in a growing economy a constant real debt-to-GDP ratio can only be stable if the real rate of interest on government debt is lower than the growth rate of real GDP which in turn leads to a violation of the government’s solvency constraint? To answer this question we will first describe Romer’s approach (Romer 2012) and how it fits into this model. Then we describe the approach by Gärtner 2009 and de Grauwe 2014 (G2), and finally we compare their approach with the one presented here.

Following Romer 2012, p. 586 we write the government’s solvency constraint in the following form:

$$\lim_{t \to \infty} PVB(t) = \lim_{t \to \infty} B(t)e^{-R_B(t)} \leq 0$$  \hspace{1cm} (106),

with

$$R_B(t) = \int_{\psi=0}^{t} r(\psi)d\psi$$  \hspace{1cm} (107),

where $PVB$ denotes the present value of public debt in $t = 0$ and $R_B(t)$ the compound interest on government bonds from $t = 0$ until $t$. For the corresponding rate of change we can write

$$\dot{PVB}(t) = \dot{B}(t) - R_B(t)$$  \hspace{1cm} (108).

From eq. (21) we get:

$$\dot{\hat{B}}(t) = \frac{d(t)}{b(t)} + f_B(t)$$  \hspace{1cm} (109)

and hence with eq. (108)

$$\dot{PVB}(t) = \frac{d(t)}{b(t)}$$  \hspace{1cm} (110).

According to eq. (109) the rate of change of public debt is the sum of two components: the rate of interest on government bonds and a wedge, namely the primary-deficit-to-public-debt ratio. If the primary deficit vanishes the rate of change of public debt just equals the rate of interest on government bonds. Whether the rate of change of public debt is smaller or larger than the rate of interest on government bonds just depends on the primary deficit being larger or smaller than zero. As public debt is positive as long as households’ propensity to invest in government bonds is positive, the present value of public debt has to converge to zero in order to satisfy the government’s solvency constraint in our model and hence, for positive values, its rate of change has to be negative when time goes to infinity. Once the present value of public debt is zero, its rate of change has to vanish for the government’s solvency constraint to be satisfied, hence:

29 This formulation is more general than the one used by Blanchard et al 1990, p. 12, who assume a constant rate of growth of real GDP and a constant real rate of interest.
\begin{align*}
\lim_{t \to \infty} \text{PVB}(t) = \begin{cases} 
\lim_{t \to \infty} \frac{d(t)}{b(t)} < 0 & \forall \text{ PVB}(t) > 0 \\
\lim_{t \to \infty} \frac{d(t)}{b(t)} = 0 & \forall \text{ PVB}(t) = 0
\end{cases} \tag{111}.
\end{align*}

According to eq. (111), public debt being positive, the primary deficit or surplus determines whether the government’s solvency constraint is satisfied or not.

Obviously, if the model economy is not in the steady state right from the start, the steady state values are only approached if the steady state is asymptotically stable. Only under this condition holds:

\begin{align*}
\lim_{t \to \infty} \text{PVB}(t) &= \text{PVB}^* \\
&= \frac{d^*}{b} = \hat{B}^* - r_B^* < 0 & \forall \text{ PVB}(t) > 0 \tag{112}.
\end{align*}

Hence, the stability of the steady state also matters for the answer to the question of whether the government’s solvency constraint is satisfied or not. In the steady state the rate of change of public debt equals the steady state growth rate of national income \((\delta + n)\) according to eq. (87). Thus from (112) follows the minimum threshold of the steady state rate of interest on government bonds \(r_B^{\text{sol}}\Min\) that has to be exceeded in order to satisfy the solvency constraint as long as the present value of public debt is positive and that has to be equalized by the rate of interest on government bonds when the present value of public debt has vanished:

\begin{align*}
r_B^{\text{sol}}\Min &= \delta + n \tag{113}.
\end{align*}

For a positive present value of public debt, fulfilling condition (112) and hence condition (106) does not only imply a strictly positive steady state primary surplus ratio (hence the right sign) for a positive steady state debt ratio according to eq. (97), but also the ‘right’ size of the steady state primary surplus ratio. If the steady state primary surplus ratio exceeded the difference between the rate of interest and the growth rate of public debt multiplied by the debt ratio, the debt ratio would eventually approach zero. If the steady state primary surplus ratio were smaller than the difference between the rate of interest and the growth rate multiplied by the debt ratio, the government’s intertemporal budget constraint would not be satisfied and the debt ratio would eventually approach infinity.\(^{30}\)

Taking into account eq. (92), condition (112) can be rewritten as:

\begin{align*}
\text{PVB}^* &= (\delta + n) \left[ 1 - \frac{(1-\sigma)(1-\alpha)}{s_{pr}^g + s_{pr}^g (1-\sigma)(1-\alpha)(1-\tau)} \right] = (\delta + n) \left( \frac{s^g_{pr}}{s_{pr}^g} - \frac{s^g_{pr} (1-\sigma)(1-\alpha)}{s_{pr}^f} \right) \leq 0 \tag{114}.
\end{align*}

According to eq. (114), for \((\delta + n) > 0\) the sign of the steady state rate of change of the present value of public debt does not depend on the size of the steady state growth rate of national income \((\delta + n)\), as

\(^{30}\) This conclusion is slightly different from Sims’ statement in his AEA Presidential Address for a static economy: “But if debt is real and the country finds itself unable to maintain primary surpluses above its predetermined real debt service commitment, it must default, even if in absolute terms it is running substantial primary surpluses.” (Sims 2013 p. 568). According to eq. (19), Sims’ statement implies an absolute decrease in public debt. In a static economy it is enough to avoid default if a country’s primary surpluses just equal its predetermined real debt service commitment, as the present value of a constant debt tends to zero if, as assumed by Sims, the rate of interest is positive. In a growing economy with a constant debt ratio both the budget deficit and the primary surplus have to be smaller than the government’s interest payments. See inequalities (99) and (100).
the latter influences the steady state values of both the rate of change of public debt and the rate of interest on government bonds in a multiplicative manner. Instead, the sign of the steady state rate of change of the present value of public debt depends on all the other parameters that influence the steady state rate of interest on government bonds alone (see the fraction in eq. (114) and taking into account eq. (92)). Considering the partial derivatives of the steady state rate of change of the present value of public debt, it is certainly not surprising that a higher tax rate helps to satisfy the government’s solvency constraint. From eq. (114) and eq. (75) follows as the minimum threshold of the tax rate \( \tau_{\text{Min}}^{\text{sol}} \) that has to be exceeded for the government’s solvency constraint to be satisfied as long as the present value of public debt is positive and that has to be equalized by the rate of interest on government bonds when the present value of public debt has vanished:

\[
\tau \geq \tau_{\text{Min}}^{\text{sol}} = 1 - \frac{(1 - \sigma)(1 - \alpha)}{s_{pr} + s_{pr}(1 - \sigma)(1 - \alpha)}; \quad \frac{\partial \tau_{\text{Min}}^{\text{sol}}}{\partial \sigma} > 0, \quad \frac{\partial \tau_{\text{Min}}^{\text{sol}}}{\partial \alpha} > 0, \quad \frac{\partial \tau_{\text{Min}}^{\text{sol}}}{\partial s_{pr}} > 0, \quad \frac{\partial \tau_{\text{Min}}^{\text{sol}}}{\partial s_{pr}^g} > 0 \quad (115),
\]

and hence with eq. (90):

\[
\tau \geq \tau_{\text{Min}}^{\text{sol}} = 1 - \frac{f_{g}b^*}{s_{pr}(1 + f_{g}b^*)} \quad (116).
\]

If the tax rate exceeds the minimum threshold there is a primary surplus in the steady state according to (98). Hence, in order to fulfill the solvency constraint it is sufficient for the government to realize a primary surplus in the steady state, as long as the steady state is stable. The minimum threshold of the tax rate increases with increases in the safety discount factor, the production elasticity of labor, and households’ saving rates (see also Figure 9).

Alternatively, for a given tax rate, from eq. (114) and eq. (75) follows as maximum threshold of the safety discount factor \( \sigma_{\text{Max}}^{\text{sol}} \) that is not to be reached for the government’s solvency constraint to be satisfied as long as the present value of public debt is positive and that has to be equalized when the present value of public debt has vanished:

\[
\sigma_{\text{Max}}^{\text{sol}} = 1 - \frac{s_{pr}(1 - \tau)}{(1 - \alpha)(1 - s_{pr}(1 - \tau))}; \quad \frac{\partial \sigma_{\text{Max}}^{\text{sol}}}{\partial \sigma} > 0, \quad \frac{\partial \sigma_{\text{Max}}^{\text{sol}}}{\partial \alpha} < 0, \quad \frac{\partial \sigma_{\text{Max}}^{\text{sol}}}{\partial s_{pr}} < 0, \quad \frac{\partial \sigma_{\text{Max}}^{\text{sol}}}{\partial s_{pr}^g} < 0 \quad (117). \]

At first sight it may seem counterintuitive that a higher safety discount factor and hence a lower steady state rate of interest rate on government bonds, a lower steady state ratio of government’s interest payments to national income, and a lower steady state level of public debt make it more difficult to satisfy the government’s solvency constraint in the long run. But a decline in the interest rate decreases the budget deficit by far less than the government’s interest payments. Hence, public consumption increases, whereas the primary surplus decreases, making it harder to pay back government debt. This result is also manifested by the steady state growth rate of government debt not being affected by a change in the safety discount factor, whereas the steady state rate of interest on government bonds is lowered.

Satisfying the liquidity constraint requires public consumption to be non-negative and this requires a sufficiently large tax rate, as the steady state ratio of public consumption to national income rises with the tax rate according to eq. (101):

\[
31 \text{ Since the outbreak of the Great Recession in many countries the central banks have lowered the level of interest rates at which banks can refinance themselves. In the majority of these countries this has led to a decline of the rate of interest for government bonds. As a first approximation, such a decline can be modeled by an increase in the safety discount factor.}
\]
From eq. (101) we see that public consumption vanishes if the following relationship holds:

\[ 1 = \left(1 - (1 - \lambda) s_{pr}^g\right)(1 - \tau) \left(1 + \frac{s_{pr}^f (1 - \alpha)(1 - \alpha)}{s_{pr}^f + s_{pr}^g (1 - \alpha)}\right) \]  

(119)

which for \( \lambda = 1 \) simplifies to

\[ 1 = (1 - \tau) \left(1 + \frac{s_{pr}^f (1 - \alpha)(1 - \alpha)}{s_{pr}^f}\right) = (1 - \tau)(1 + r^*_g b^*) \]  

(120).

With the Golden Rule of Public Finance being applied, public consumption is equal to zero according to eq. (120) if the impacts of being taxed and of receiving interest income from the government just cancel out, i.e. if private households’ available income equals national income. In this case the households’ propensities to consume and invest with respect to available income obviously coincide with their propensities to consume and invest with respect to national income. Hence, if all savings invested in government bonds are used to finance public investment no space is left for public consumption.

From eq. (101) we can derive the minimum tax rate for which the liquidity constraint is fulfilled:

\[ \tau_{\text{Min}}^{\text{liq}} = 1 - \frac{s_{pr}^f}{\left(1 - (1 - \lambda) s_{pr}^g\right)\left(s_{pr}^f + s_{pr}^g (1 - \alpha)(1 - \alpha)\right)} \]  

(121),

and hence with eq. (90):

\[ \tau_{\text{Min}}^{\text{liq}} = 1 - \frac{1}{\left(1 - (1 - \lambda) s_{pr}^g\right)\left(s_{pr}^f + r^*_g b^*\right)} \]  

(122),

i.e. the tax rate that just generates enough tax revenue in order to finance the government’s interest payments if \( \lambda = 1 \), i.e. if the Golden Rule of Public Finance is followed. Furthermore we can compute:

\[ \tau_{\text{Min}}^{\text{liq}} \left[\begin{array}{c}
lambda = \frac{s_{pr}^f - \left(1 - s_{pr}^g\right)(1 - \alpha)}{s_{pr}^f + s_{pr}^g (1 - \alpha)(1 - \alpha)}
\end{array}\right] = 0 \]  

(123).

In this case the liquidity constraint would be satisfied with a zero tax rate, yet government’s interest payments would be financed from the budget deficit – an obvious case of Ponzi finance, and hence of insolvency. In order to satisfy both the liquidity and the solvency constraint, and hence two out of three sustainability conditions, the tax rate has to be larger than the larger of the two minimum thresholds (see also Figure 9):

\[ \tau^{\text{sus}} \geq \tau_{\text{Min}} = \max\left(\tau_{\text{Min}}^{\text{liq}}, \tau_{\text{Min}}^{\text{sol}}\right) \]  

(124).

32 As will be shown below, the stability conditions do not depend on the tax rate in the case under consideration (see inequalities (142) and (143)).
By setting eq. (101) equal to zero and solving for the safety discount factor a minimum threshold can be established at which public consumption vanishes:

$$\sigma_{\text{Min}}^{\text{liq}} = 1 + \frac{s_f^{pr}}{(1-\alpha)s_f^{pr}} \left[ 1 - \frac{1}{(1-s_f^{pr}(1-\lambda))(1-\tau)} \right]; \frac{\partial \sigma_{\text{Min}}^{\text{liq}}}{\partial \tau} < 0, \frac{\partial \sigma_{\text{Min}}^{\text{liq}}}{\partial \alpha} < 0, \frac{\partial \sigma_{\text{Min}}^{\text{liq}}}{\partial \lambda} > 0; \frac{\partial \sigma_{\text{Min}}^{\text{liq}}}{\partial s_f^{pr}} > 0 \quad (125).$$

In order to fulfil both the liquidity and the solvency constraint, and hence to be sustainable, the safety discount factor has to satisfy the following condition:

$$\sigma_{\text{Min}}^{\text{liq}} < \sigma_{\text{Max}}^{\text{sus}} < \sigma_{\text{Max}}^{\text{sol}} \quad (126).$$

For $\delta + n > 0$ the solvency constraint, i.e. inequality (114) holds if

$$\left[ \frac{B}{Y} \right] = \frac{s_g^{pr}}{s_f^{pr}} (1-\alpha)(1-\lambda) = \frac{\gamma^*}{\delta + n} \quad (127),$$

i.e. if the steady state public-deficit-to-national-income ratio is smaller than the steady state government-interest-payments-to-national-income ratio. According to inequality (127) the steady state government-interest-payments-to-national-income ratio is the upper threshold of the steady state public-deficit-to-national-income ratio $\left[ \frac{B}{Y} \right]$ that is not to be reached if the government’s solvency constraint is to be satisfied. This upper threshold is the lower, the lower is the propensity to invest in public bonds compared to the propensity to invest in private bonds/shares, the lower is capital’s share in national income, and the higher is the safety discount factor.

But, as Figure 7 (a) shows, by setting a tax rate that exceeds the minimum tax rate according to inequality (115) the government has an instrument to make sure that its solvency constraint is satisfied, even if the safety discount is large.$^{33}$

For the corresponding upper threshold of the steady state debt ratio follows from inequality (127):

$$b^* = \frac{s_g^{pr}}{s_f^{pr}}(1-\alpha)(1-\lambda) = \frac{\gamma^*}{\delta + n} = \frac{b^*}{\delta + n} \quad (128).$$

(a) Deficit ratios: steady state value and its upper threshold

(b) Debt ratios: steady state values and their upper thresholds

Figure 7: Steady state values of the deficit and the debt ratio as well as their upper thresholds

($s_g^{pr} = 0.02; s_f^{pr} = 0.1; \gamma = 1; \alpha = 0.7; \beta = 0.2; n = 0; \delta = 0.02; \lambda = 1$) Exozupi: x:1.2, y:3.1, z: 1.1

$^{33}$ Assuming the labor supply to depend on the tax rate might change this conclusion.
From Figure 7 (b) we see, among others, the steady state growth rate and the safety discount factor determine the upper threshold of the steady state debt ratio. Summing up, from (99), (112), (127), and (128) follows

\[
\begin{align*}
\left( \frac{B}{Y} \right)^* & = \left( \frac{\dot{B}}{Y} \right)^* \\
0 & \leq \delta < \gamma \\
(\text{PVB}) & \iff b^* \geq b^*
\end{align*}
\]

(129).

Efficiency of public debt requires the shadow rate of return on the public deficit \( (s \tau_B) \) to be greater than or equal to the rate of interest on government bonds:

\[
sr_B = \lambda Y_{\tau g} = \lambda (1 - \alpha - \beta) \kappa_{pr}^B k_{pr}^{\alpha - \beta} \geq \tau_B
\]

(130).

Figure 8 shows a case where the shadow rate of return on the public deficit exceeds the rate of interest on government bonds which in turn exceeds the rate of growth of public debt in the steady state. Hence condition (130) is fulfilled in the steady state. The steady state level of public debt is efficient. Even a slightly higher public debt due to a sufficiently small increase in the households’ propensity to invest in government bonds would be efficient according to Figure 8:

![Figure 8: Rate of interest, shadow rate of return on the public deficit, and growth rate of public debt](image)

According to eqs (57) and (90) public debt is efficient in the steady state if the so-called spread factor \( 1 - \sigma \) satisfies the following condition:

\[
1 - \sigma \leq (1 - \sigma)^{\text{Max}} = \frac{(1 - \alpha - \beta) s_{pr}^f}{(1 - \alpha) s_{pr}^g}
\]

(131).

The light gray area in Figure 9 shows the set of combinations of the tax rate and the spread factor for which conditions (124), (126), and (131), and hence both the liquidity and the solvency constraint as well as the efficiency condition are fulfilled. Using part of the budget deficit for public consumption \( (\lambda = 0.5) \) allows the government to stay liquid at a lower tax rate for a given spread factor (see Figure 9 (b)).

34 The parameter values in the various figures are based on rough estimates for Germany.

35 The horizontal axis of Figure 9 shows the spread factor \( 1 - \sigma \) in order to facilitate the visual comparison with the cases of an exogenously fixed interest rate in section 6.
According to Gärtner 2009 p. 395f and de Grauwe 2014 annex (G2), the government’s solvency condition leads to instability of the steady state debt ratio and hence to non-sustainability of public debt. Thus, let us have a closer look at their approach.\(^{36}\)

By differentiating the debt ratio with respect to time (see eq. (47)), we can compute

\[
\dot{b} = \frac{\dot{Y} - \dot{b}}{\frac{\dot{B}}{Y} - \dot{Y}} = \frac{\dot{B} - \dot{Y}}{\dot{Y}}
\]

(132).

The change of the debt ratio is a function of the private and the public-capital-to-labor ratio which is shown in Figure 10 (a). The larger the public-capital-to-labor ratio and the smaller the private capital-to-

\(^{36}\) See in this context Greenlaw et al. 2013, too.
labor ratio, the smaller is the change of the debt ratio and the larger is the debt ratio. The change of the debt ratio becomes negative at large debt ratios which is a clear sign of the stability of the steady state.

Figure 10 (c) shows that the public deficit ratio does not vary much with the private and the public-capital-to-labor ratio, whereas the public interest payments-to-national income ratio does, as does the primary deficit ratio, in the opposite direction.

By inserting eq. (16) into eq. (132) and taking into account the definition of the primary budget deficit the following differential equation for the debt-to-national-income ratio can be derived:

\[ \dot{b} = \frac{C_g + I_g - T}{Y} - (\dot{Y} - r_B)b = d - (\dot{Y} - r_B)b \quad (133), \]

from which – by setting \( \dot{b} = 0 \)

\[ b^* = \frac{d}{\dot{Y} - r_B} \quad (134), \]

which is just the same equation as eq. (97) if \( d, \dot{Y}, \) and \( r_B \) are equal to their steady state values. In principle, \( d, \dot{Y}, \) and \( r_B \) could be functions of \( b \). Hence, by linearizing eq. (133) around the equilibrium by means of a Taylor series of the first order we get:

\[ b^T = d(b^*) - (\dot{Y}(b^*) - r_B(b^*))b + d\dot{b}(b^*) (b - b^*) - b^* \left( \ddot{Y}_b(b^*) - (r_B(b^*)) \right)(b - b^*) \quad (135), \]

where \( T \) refers to Taylor and \( x\dot{b}(b^*) = x^* \) and \( (x)_{\dot{b}}(b^*) = x^{**} \) \( \forall x = d, \dot{Y}, r_B \), and by inserting the first derivative of eq. (22):

\[ d^{**} = -(1 - s_{pr}^g (1 - \tau)) r_B - r_B^* \left( 1 - s_{pr}^g (1 - \tau) \right)b^* \quad (136) \]

into eq. (135) we can derive:

\[ b^T = d^* - (\dot{Y}^* - r_B^*)b - r_B^* \left( 1 - s_{pr}^g (1 - \tau) \right)(b - b^*) - b^* \left[ \ddot{Y}_b^* - r_B^* \left( s_{pr}^g (1 - \tau) \right) \right](b - b^*) \quad (137). \]

According to eq. (137) the steady state debt ratio is stable if

\[ b^{T**} = -\dot{Y}^* (1 + e_{Y,b}^*) + r_B^* \left( 1 + e_{r_B,b}^* \right) s_{pr}^g (1 - \tau) < 0 \quad (138), \]

where \( e_{Y,b} \) denotes the elasticity of the rate of growth of GDP with respect to the debt ratio, and \( e_{r_B,b} \) the elasticity of the rate of interest on government bonds with respect to the debt ratio. Whereas \( e_{Y,b} \) is close to zero, \( e_{r_B,b} \) is close to minus unity, largely because of the assumption of a Cobb-Douglas production function. Furthermore, \( s_{pr}^g (1 - \tau) \) is comparatively small. Hence, condition (138) may be fulfilled even if the rate of interest on government bonds exceeds the growth rate.

G2 treat \( d, \dot{Y}, \) and \( r_B \) as exogenous parameters. Hence they (implicitly) assume:

\[ d^* = \dot{Y}^* = r_B^* = 0 \quad (139). \]

Accordingly, G2’s (linear) differential equation for the debt ratio can be written as follows:

---

37 de Grauwe calls this the necessary condition for maintaining solvency (de Grauwe 2014). Hence, his use of the term ‘solvency’ is different from mine.
where \( G \) refers to \( G_2 \). By inspecting eq. (140) we see that the differential equation stipulated by \( G_2 \) is linear, and intersects the vertical axis (\( b^G \)) at \( d^* \) and the horizontal axis (\( b^G \)) at \( b^* = b^{G*} \). If \( d^* < 0 \) and hence \( \dot{Y}^* < r_b^* \) the resulting differential equation \( b^G \) is positively sloped and, thus, the steady state debt ratio is unstable under the special conditions (139), but is stable if, as in the model presented above, the primary deficit ratio, the growth rate of national income and especially the rate of interest are allowed to vary, as will be shown in the following.

All markets are assumed to be in equilibrium even outside the steady state in the model presented above. Hence, we are dealing with stability in the restricted sense of equilibrium dynamics, not disequilibrium dynamics (Hahn and Matthews 1964, p.782 and pp 804 ff.). By linearizing the differential equations (62) and (63) in the relevant steady state we can compute the discriminant \( \Delta \):

\[
\Delta = (1-\alpha)^2(\delta+n)^2 > 0 \quad \forall \ 0 < \alpha < \frac{\delta+n}{\alpha}
\]

Hence, we have two real roots. For the trace \( tr \) of the Jacobian matrix we get:

\[
tr = -(1+\alpha)(\delta+n) < 0 \quad \forall \ \alpha > 0; \ (\delta+n) > 0
\]

and for the determinant \( det \) of the Jacobian matrix:

\[
det = \alpha(\delta+n)^2 > 0 \quad \forall \ \alpha > 0; \ (\delta+n) \neq 0
\]

Because of eqs (141) - (143) the relevant steady state is a stable node (see Gandolfo 2010, p.358), as long as \( (\delta+n) > 0 \) and \( 1 > \alpha > 0 \). This result is in line with the stability analysis of the differential equation for the aggregate-capital-labor-ratio along the SSL (see eq. (82)), whose slope in the steady state is:

\[
\frac{d\kappa^{SSL}}{d\kappa^{SSL}} = -\alpha(\delta+n) < 0 \quad \forall \ \alpha > 0; \ (\delta+n) > 0
\]

Remarkably, the stability conditions are essentially the same as in the original Solow model. Hence, for stability it does not make a difference whether households’ savings are invested in one or several types of capital, fully or partly, as long as the saving rates and corresponding investment rates are positive and exogenous. The stability of the economically relevant steady state just depends on the sum of the growth rates of technological progress and labor supply as well as on the partial production elasticity of labor in efficiency units.

A look at the stream plot in Figure 11 shows that many trajectories lead to the steady state, depending on the initial conditions. A second look shows that the (stable) steady state locus (SSL) determined by eq. (64) is not crossed by any of the trajectories, as the rates of change of the private and the public capital-to-labor ratio are equal (\( \dot{\kappa}_{pr} = \dot{\kappa}_{g} \)) on the SSL. Hence, a trajectory that starts on the SSL remains on the SSL.

---

38 At first sight, one might even argue that, according to the inequalities (141) - (143), the equilibrium is a stable node for \( \alpha > 1 \), too. But this would not take into account the fact that the differential equations were derived under the assumption that the aggregate production function is linear homogenous.

39 How stability is affected if investment functions, that are independent from saving, are introduced in the Solow model is treated in Nikaido 1980. However, this is a study of disequilibrium dynamics in the sense of Hahn and Matthews 1964.
Figure 11: Stream plot of private and public capital-to-labor ratios

\[
\begin{align*}
&\gamma = 0.7; \alpha = 0.2; \tau = 0.32; n = 0; \delta = 0.02; \sigma = 0.6; \lambda = 1 \\
&\kappa_{pr} = 0 \quad \text{and} \quad \kappa_{g} = 0 \quad \text{isoclines whose intersection determines the steady state.}
\end{align*}
\]

Figure 12 (a), (c), and (e) show the phase diagram of private and public capital-to-labor ratios with, among others the \(\kappa_{pr} = 0\) and \(\kappa_{g} = 0\) isoclines whose intersection determines the steady state. In addition to the two isoclines, Figure 12 (a), (c), and (e) show the combinations of private and public capital-to-labor ratios for which \(d = d^*\), \(\dot{Y} = \dot{Y}^*\), and \(f_B = f_B^*\). Obviously, these combinations do not coincide outside the steady state. When \(b \neq b^*\) the \(d = d^*\)-locus and the \(f_B = f_B^*\)-locus cannot coincide, as, according to eq. (22), a constant primary deficit-to-national-income ratio implies a constant public interest payments-to-national-income ratio. Hence, if the debt ratio increases by one percent the rate of interest on government bonds has to decrease by one percent.

To simultaneously postulate \(d = d^*\), \(\dot{Y} = \dot{Y}^*\), and \(f_B = f_B^*\) outside the steady state, as G2 do, may lead to misleading results, especially if \(f_B^* > \dot{Y}^*\), as will be shown below. The \(d = d^*\)-locus coincides with the steady state locus (see eq. (95)). On it not just the primary deficit-to-national-income ratio equals its steady state value, but all the variables whose value is determined by the ratio of private to public capital, especially the budget deficit-to-national-income ratio and the public interest payments-to-national-income ratio.

In order to compare the approach of G2 more directly with ours we now derive the differential equation for the debt-to-national-income ratio from the model presented above where the number of new government bonds traded is determined by the demand of private households (see eq. (23)). By inserting eq. (23) into eq. (133) the following differential equation for the debt-to-national-income ratio can be derived:

\[
b = s_{pr}^b (1 - \tau) - \left( \dot{Y} - s_{pr}^b (1 - \tau) f_B \right) b \tag{145}
\]

Let us now have a look at the differential equation for the public debt-to-national-income ratio if we only postulate one of the three conditions \(d = d^*\), \(\dot{Y} = \dot{Y}^*\), and \(f_B = f_B^*\) to hold at a time. We start with \(d = d^*\) and hence on the SSL. From eq. (145) we get

\[
b = s_{pr}^b (1 - \tau) (1 + f_B b) - \dot{Y} b \tag{146}
\]
According to eq. (90) the ratio of public interest payments to national income is constant along the SSL. Hence, on the steady state locus the rate of interest has an influence on the intercept of the differential equation for the debt ratio, not its slope, and thus not on stability. Furthermore, we can compute the growth rate of national income along the steady state locus (see Figure 13 (a)):

\[
\dot{Y}^{SSL} = \alpha(\delta + n) + (1 - \alpha) \frac{b^{SSL}}{\dot{g}_{pr}^{SSL}} \frac{\partial \dot{Y}^{SSL}}{\partial \sigma} < 0
\]  

as well as the growth rate of public debt.
\[
\dot{B}^{SSL} = \frac{\dot{s}^{g}}{b^{SSL}} = \frac{s^{g}_{pr}(1-\tau)}{b^{SSL}} + s^{g}_{pr}(1-\tau)\dot{B}^{SSL}; \quad \dot{B}^{SSL} > 0 \quad \forall \quad \dot{s}^{g}_{pr}, \dot{b}^{SSL} > 0; \quad \frac{\dot{B}^{SSL}}{\dot{b}^{SSL}} < 0 \quad (148),
\]

which changes by a small fraction in the same direction whenever the interest rate changes. According to eqs (147) and (148), along the steady state locus the growth rates of national income and public debt are the lower, the higher the debt ratio. Both growth rates fall with an increase in the safety discount factor. From eq. (148) we get

\[
\dot{B}^{SSL} = \dot{\gamma}^{SSL} - \dot{\beta}^{SSL} \quad (149),
\]

and hence with eq. (147)

\[
\dot{B}^{SSL} = \alpha(\delta + n) - \alpha\dot{B}^{SSL} \quad (150),
\]

which is shown in Figure 13 (b). Whereas Figure 13 (a) just shows how the steady state growth rate of public debt can be determined, Figure 13 (b) shows its stability, too.

The first term on the right hand side of eq. (147) enters into the slope of the differential equation for the debt ratio and, hence, matters for stability. By using eqs (90), (146) and (147) we can finally derive the following differential equation for the debt-to-national-income ratio on the SSL:

\[
\dot{b}^{SSL} = \alpha\dot{s}^{g} - \alpha(\delta + n)b^{SSL} \quad (151).
\]

From eq. (151) we can directly compute the steady state value of the public-capital-to-labor ratio which obviously leads to the same result as eq. (88). According to eq. (151) the steady state debt-to-national-income ratio is stable if \(0 < \alpha < 1\) and \(\delta + n > 0\). These, again, are the same conditions that were derived for the two-dimensional system of differential equations above for the steady state to be a stable node (see eqs (141) - (143)).

![Figure 13: Steady state growth rate of public debt and its stability](a) SSL: growth rate of national income as function of growth rate of public debt (b) SSL: growth rate of growth rate of public debt as function of growth rate of public debt

Figure 12 (a) shows how the steady state is approached along the steady state locus if the starting point is located on the steady state locus. For the same case we can see from Figure 12 (b) that the green line which shows the computed combinations of the debt-to-national-income ratio (b) and its change in time (\(\dot{b}\)) coincides with the red dashed line, which shows the differential equation (151). For the steady state locus we can compute the elasticity of the rate of interest on government bonds and the growth rate of national income with respect to the debt ratio from eqs (90) and (147):

\[
\frac{\dot{b}_{SSL}}{b_{SSL}} = -1 \quad (152),
\]
Hence, the slope of the Taylor approximation according to eq. (138) coincides with the slope of the phase curve of the debt ratio along the steady state locus according to eq. (151).

If however the starting point is either located on the \( \hat{Y} = \hat{Y}^* \)-locus or on the \( r_B = \hat{r}_B \)-locus, the trajectories do not follow the respective locus (see Figure 12 (c) and (e)). From Figure 12 (d) and (f) we observe that the green line showing the computed combinations of the debt-to-national-income ratio \( b \) and its change in time \( \dot{b} \) is located between the red-dashed differential equation for the steady state locus and the blue line which represents a linearized (hence superscript l) differential equation for the debt ratio. The latter can be used as an approximation for starting points that are not in the vicinity of the steady state locus. We obtain this linearized differential equation for the debt ratio from eqs (92), (137) and (145), by using the steady state values of \( \hat{Y} \) and \( r_B \) and assuming \( \dot{Y}_b(b^*) = (\dot{b}^*_B)_b(b^*) = 0 \) as a first approximation:

\[
\dot{b}^l = s_{pr}^g (1 - \tau) - \frac{(\delta + n) s_{pr}^g}{s_{pr}^g + (1 - \sigma)(1 - \alpha)} b^l
\]

The linearized differential equation for the debt ratio is equivalent to the Taylor approximation according to eq. (135) for \( d_b(b^*) \neq 0 \) and \( \dot{Y}_b(b^*) = (\dot{b}^*_B)_b(b^*) s_{pr}^g (1 - \tau) = 0 \). Obviously, for \( 0 < \alpha, \sigma, \tau, s_{pr}^g, s_{pr}^f < 1 \), according to eq. (154) the resulting equilibrium debt ratio is stable, as long as \((\delta + n)\) is positive, the same result as derived above (see inequalities (141) - (143)). From eq. (145) we see that for the equilibrium debt ratio to be stable the growth rate of national income does not have to exceed the rate of interest on government debt, but just the tiny fraction \( s_{pr}^g (1 - \tau) \) of it. From Figure 12 (d) and (f) we see that the green line showing the computed combinations of the debt-to-national-income ratio \( b \) and its change in time \( \dot{b} \) is closer to the blue line \( \dot{b}^l \) if the starting point is located on or in the vicinity of the \( \hat{Y} = \hat{Y}^* \)-locus than on or in the vicinity of the \( r_B = \hat{r}_B \)-locus. This is due to \( \hat{Y}^* \) entering into the slope of the blue line \( \dot{b}^l \) with a weight of one, and \( r_B^* \) only with a tiny weight of \( s_{pr}^g (1 - \tau) \). Hence, the impact of the – with respect to the underlying model – inconsistent assumption \( r_B = r_B^* \) in addition to \( \hat{Y} = \hat{Y}^* \) is small enough not to change the qualitative result with respect to the steady state’s stability. In any case, the slope of \( \dot{b} \), \( \dot{b}^l \), and \( \dot{b}^{SSL} \) is negative, correctly indicating stability of the steady state, whereas the slope of the black line \( \dot{b}^G \) is positive if \( \hat{Y}^* < r_B^* \), incorrectly indicating instability of the steady state, as the further incompatible assumption \( d = d^* \) is made.

Another possible approximation is to stipulate that some of the differential equations are continuously equal to zero. This can be justified if some variables adjust to shocks faster than others (see Haken 1977, pp 202ff., Zhang 1991, pp 193ff.). As the dynamic model presented just consists of two differential equations and as the focus of interest is the development of public debt which is largely determined by the public-capital-to-labor ratio, in the following we consider the case \( \kappa_{pr} = 0 \) as one more possible
approximation. Here, the development is assumed to follow the $\kappa_{pr} = 0$-isocline which is in the vicinity of the $r_B = r_B^*$-locus (see Figure 12 (a), (c), and (e)). The advantage of this approximation is its consistency with the underlying model outside the steady state. The corresponding phase curve for the debt ratio $b(\kappa_{pr} = 0)$ is shown in Figure 12 (b), (d), and (f), too. Its slope is negative, correctly indicating the stability of the steady state.

Thus we come to the following conclusion: In a stable steady state the rate of interest on government debt may exceed the growth rate of national income and hence the government’s solvency constraint may be satisfied. Thus the answer to the question whether in general a constant debt-to-GDP ratio can only be stable in a growing economy if the growth rate of real GDP and hence real government debt exceeds the real rate of interest on government debt, which in turn leads to a violation of the government’s solvency constraint, is no. This implies that stability condition (138) can be fulfilled.

Figure 14: Fixed tax rate: relations along the $\kappa_{pr} = 0$-isoeline and steady state locus (ImplotExozupi)

From eqs (22), (87), and (92) we can finally derive the following linearized relation of the primary deficit ratio ($d^1$) and the debt ratio for starting points on or in the vicinity of the $\tilde{Y} = \tilde{Y}^*$-locus:

$$d^1 = s^g_{pr} (1-\tau) - \frac{(1-s^g_{pr}(1-\tau))(\delta + n)(1-\sigma)(1-\alpha)}{(1-\tau)(s^f_{pr} + s^g_{pr}(1-\sigma)(1-\alpha))}b^1$$

---

40 As will become clear, to assume $\kappa_{pr} = 0$ instead of $\dot{\kappa}_w = 0$ can especially be justified in the case of a primary surplus ratio target which is considered in section 5 below. Hence, to make comparisons easier, this assumption will also be made in the cases of a fixed tax rate and a budget deficit ratio target.
According to eq. (155), for a constant rate of interest on government debt at its steady state level \( f_B^* \), an increase in the debt ratio implies a fall in the primary deficit ratio in order to satisfy the government’s liquidity constraint. According to Figure 14 (b), along the \( \kappa_{pr} = 0 \)-isocline the primary deficit ratio also falls with an increasing debt ratio, as the rate of interest is almost constant. In Figure 14 (b) and (d) the steady state debt ratio is determined by the intersection of the curves representing the actual budget deficit ratio \( (B/Y)^* \) and the equilibrium budget deficit ratio \( (B/Y)^{eq} = (\delta + n)b \).

Now, let us consider the question whether – outside the model presented – there are circumstances under which \( \delta_B(b^*) = 0 \) holds even if the tax rate is fixed and if the starting point is not located on the \( d = \delta^* \)-locus (SSL). This is the case if the demand for government bonds is infinitely elastic and the government chooses to keep the primary deficit ratio constant. In this case, the debt dynamics are determined by the financial needs of government, whereas in the model presented here the public debt dynamics are determined by the demand of private households for government bonds. In the latter case public expenditures on goods and services are adjusted to satisfy the liquidity constraint if the government’s interest payments are changing with public debt, whereas this is not the case with the former. Here it is assumed that the government can keep the ratio of the primary deficit to national income constant even if its interest payments increase. In the former case it is assumed that an increase in public interest payments leads to an increase in the public deficit by the same amount. In the latter case an increase in public interest payments just leads to an increase in the demand for new government bonds according to the marginal propensity to invest in government bonds after tax \( s_{pr}^g (1 - \tau) \).

The lion’s share of an increase in public interest payments \( (1 - s_{pr}^g (1 - \tau)) \) has to be compensated by a decrease in the ratio of the primary deficit to national income. Hence, the positive feedback and hence destabilizing effect of an increase in the interest rate on public debt on the dynamics of public debt is much lower in the presented model than in the models by G2.

Public deficits financed exclusively by issuing new government bonds cannot exceed the demand for newly issued government bonds. In principle, budget deficits could be lower than the demand by households. But in the model presented here it is assumed that the government maximizes its budget deficit given households’ demand for newly issued government bonds. As shown above, this does not create problems for sustainability, as long as households’ propensity to invest in government bonds is constant and as long as the safety discount factor is low enough such that the government’s solvency constraint remains satisfied. Instead, in the models of G2 the public deficit is determined by the government’s decisions on spending and taxing, either without adequately considering the demand side in the market for newly issued government bonds or by unrealistically assuming that the demand for government bonds is infinitely elastic. In this case there would never be a sovereign debt crisis: An ever increasing debt ratio and correspondingly increasing public deficit ratio would not create any problems, as the increasing supply of newly issued government bonds would just be absorbed by investors.

The size of the impact of a change of the debt ratio on the primary deficit ratio is to be determined empirically. Fortunately, there exists a growing body of empirical literature on so-called fiscal reaction functions (see e.g. Greiner et al. 2007, Bohn 2008, Mendoza and Ostry 2008, Ghosh et al. 2013, Weichen-
rieder and Zimmer 2014, Cevik and Teksoz 2014). Even if the notion ‘fiscal reaction function’ indicates that the authors seem to assume that the budget deficit and hence the primary deficit is mainly determined by the government, not by the demand for newly issued government bonds, the estimates of the size of the impact of a change of the (lagged) debt ratio on the primary deficit ratio can be used nonetheless, as they are just based on correlations, not on causalities. Greiner et al. 2007 generate statistically significant estimates for France, Germany, Italy, Portugal, and the USA for the decades before 2003 in the range of about -0.15, a value that makes sure that inequality (138) is fulfilled. Based on a panel dataset covering the period 1990–2012, and hence the Great Recession as well as the Euro Crisis, for 49 advanced and emerging market economies, Cevik and Teksoz 2014 generate statistically significant estimates of the size of the impact of a change of the lagged debt ratio on the primary deficit ratio which are much smaller in absolute terms, namely around -0.01. Hence, inequality (138) is not fulfilled if the rate of interest on government debt exceeds the GDP growth rate by just more than one percentage point. Thus, it can be argued that the softening of the government’s liquidity constraints in various countries due to purchasing programs of government bonds by Central Banks, which helped to increase the difference between the GDP growth rate and the rate of interest on government bonds, also shows up in these estimates. Another explanation for the lower absolute level might be that the trajectories of private and public capital-to-labor ratios are closer to the steady state locus in more recent years.

<table>
<thead>
<tr>
<th>Government’s liquidity constraint</th>
<th>Stability conditions</th>
<th>Government’s solvency constraint</th>
<th>Long term sustainability conditions</th>
<th>Efficiency condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Endogenous rate of interest: Fixed tax rate</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
\tau \geq \frac{1 - \tau_{\text{Min}}^s}{1 - \left(1 - \lambda \right)s_{\text{pr}}} \left[ \frac{\alpha > 0 \cup (\delta + n) > 0}{\left(1 - (\alpha + \beta)\right)s_{\text{pr}}} \right]
\]

\[
\tau \geq \frac{1 - \tau_{\text{Min}}^s}{1 - (1 - \sigma)(1 - \alpha)} \left[ \frac{s_{\text{pr}} \left( \frac{\alpha > 0}{\left(1 - (\alpha + \beta)\right)s_{\text{pr}}} \right)}{\left(1 - \sigma\right)s_{\text{pr}}} \right]
\]

\[
1 > \tau \geq \max \left( \frac{\tau_{\text{Min}}^s}{1 - \sigma} \right) \left[ \frac{\alpha > 0}{(1 - \sigma)s_{\text{pr}}} \right]
\]

\[
1 - \sigma \leq \left(1 - \sigma\right)_{\text{Max}} = \frac{\alpha > 0}{(1 - \sigma)s_{\text{pr}}}
\]

Table 1: Conditions for liquidity, stability, solvency, long term sustainability, and efficiency of public debt

According to (142) and (143) stability does not depend on the parameters that matter for government’s liquidity and solvency like households’ propensities to save, the tax rate, the safety discount (or spread) factor and the ratio of public investment to budget deficit, as long as the ratio of aggregate investment to national income is positive. This means that the steady state would be stable even if the tax rate were smaller than \(\tau_{\text{Min}}^s\), hence if the liquidity constraint were violated. Thus, stability of the steady state is not sufficient to satisfy the liquidity constraint. But, e.g. it is necessary in order to avoid the violation of the liquidity constraint during the process away from the steady state in case of its instability once a shock leads to an increase of the public-capital-to-labor ratio, the debt-to-GDP ratio and the ratio of public interest payments to GDP beyond their steady state values which then leads to further increases due to positive feedback.\(^\text{44}\) Furthermore, there exists a set of combinations of the tax rate and the spread factor for which the solvency constraint is not fulfilled, but the steady state is stable nonetheless. Table 1

\(^{44}\) See below the case of an exogenous rate of interest on government bonds exceeding the steady state growth rate when government targets a primary surplus-to-GDP ratio.
summarizes the conditions for liquidity, solvency, stability, long term sustainability, and efficiency of public debt.\textsuperscript{45}

Having found more than one definition of sustainability of public debt dynamics in the literature, the question arises what their relationship is. As the solvency constraint refers to the limit of the present value of public debt when time goes to infinity this is certainly a long term concept of sustainability. Instead, stability of the steady state refers to the medium term, and the liquidity constraint to the short term. Even if the sixty percent threshold of the Maastricht treaty for the public debt ratio was not derived from any model, but was just the average of debt ratios in those countries that were considered probable member states of the European Monetary Union in the beginning of the 1990s, it has gained a certain prominence in public discussions, at least in Europe. Hence, it can be used to define subcategories even if this is not explicitly done in Table 2 (but see Figure 15 and Figure 16).

<table>
<thead>
<tr>
<th>Steady state public debt dynamics</th>
<th>Government's liquidity constraint</th>
<th>Stability conditions</th>
<th>Government's solvency constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short term unsustainable</td>
<td>violated in the steady state</td>
<td>violated or satisfied in the steady state</td>
<td>violated or satisfied in the steady state</td>
</tr>
<tr>
<td>Short term sustainable yet medium term unsustainable</td>
<td>satisfied in the steady state</td>
<td>violated in the steady state</td>
<td>violated or satisfied in the steady state</td>
</tr>
<tr>
<td>Medium term unsustainable</td>
<td>violated in the steady state</td>
<td>violated in the steady state</td>
<td>violated or satisfied in the steady state</td>
</tr>
<tr>
<td>Medium term sustainable yet long term unsustainable</td>
<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
<td>violated in the steady state</td>
</tr>
<tr>
<td>Long term sustainable</td>
<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
<td>satisfied in the steady state</td>
</tr>
</tbody>
</table>

Table 2: Categories of sustainable and unsustainable public debt dynamics

There exists the following hierarchy: for the steady state to be long term sustainable it must also be both short and medium term sustainable; for the steady state to be medium term sustainable it must also be short term sustainable. Or to put it differently: medium term sustainability requires short term sustainability, whereas short term unsustainability implies both medium and long term unsustainability, and medium term unsustainability implies long term unsustainability.

In the presented model the steady state is stable under very general conditions. Hence, the public debt dynamics are medium term sustainable if the liquidity constraint is satisfied. In Figure 15 and Figure 16 we see the dynamic equilibrium processes leading to the stable steady state for two parameter constellations which only differ in the values of households’ propensity to invest in government bonds, the safety discount factor, and in the initial values of private and public capital per capita.

The values of households’ propensity to invest in government bonds are selected in such a way that the steady state value of the public debt ratio lies above the Maastricht criterion of 60 % in Figure 15, and below in Figure 16, whereas the values of the safety discount factor in such a way that the steady state value of the rate of interest on public debt is higher than the steady state value of the growth rate of public debt in Figure 15, and lower in Figure 16. In Figure 15 the initial value of the public debt ratio is above its steady state value, in Figure 16 below. In both figures the starting points of the trajectories of the private and the public capital-to-labor ratios are not in the vicinity of one of the \( \frac{\gamma}{\gamma} = \gamma^*, \hat{Y} = \hat{Y}^*, \) and \( f_B = f_B^* \)-loci.

\textsuperscript{45} See in this context also Figure 9 in which the stability conditions are satisfied. Hence, the light gray area shows combinations of the tax rate and the spread factor for which steady state public debt is long term sustainable and efficient.
Hence, in Figure 15 we consider the case of long term sustainable public debt dynamics with increasing debt ratio, steady state primary surplus and hence satisfied government’s solvency constraint, as well as violated Maastricht debt ratio criterion, in Figure 16, instead, the case of medium term Maastricht sustainable public debt dynamics with decreasing debt ratio, steady state primary deficit and hence violated government’s solvency constraint, as well as satisfied Maastricht debt ratio criterion.
Figure 16: Medium term Maastricht sustainable but long term unsustainable public debt dynamics with decreasing debt ratio

\[
\begin{align*}
\kappa_p(0) = 1.1\kappa_p^*; \kappa_g(0) = 1.6\kappa_g^*; s_p^0 = 0.012; s_g^0 = 0.4; t = 0.7; \beta = 0.2; \tau = 0.25; \gamma = 0.02; \sigma = 0.9; \lambda = 1
\end{align*}
\]

In graph (e) of both figures we see how the primary-deficit-to-national-income ratio and the government-interest-payments-to-national-income ratio change relatively strongly compared to the almost unchanged deficit-to-national-income ratio during the adjustment process. According to eq. (23), for a giv-
en propensity to invest in government bonds and a given tax rate, (i) the deficit-to-national-income ratio can only vary if the government-interest-payments-to-national-income ratio varies, and (ii) the deficit ratio varies much less than the government-interest-payments-to-national-income ratio because of:

\[
\frac{\partial \hat{B}/Y}{\partial \hat{b}} = s_{pr}^g (1 - \tau) \ll 1
\]  

(156).

From eq. (22) we get

\[
\frac{\partial \hat{d}}{\partial \hat{b}} = - \left(1 - s_{pr}^g (1 - \tau)\right)
\]  

(157).

Hence, (i) the primary-deficit-to-national-income ratio and the government-interest-payments-to-national-income ratio move in opposite directions, and (ii) their changes are of almost equal absolute size.

At first sight one might think that the public debt dynamics look much better in Figure 16 than in Figure 15, as the debt ratio declines in Figure 16 (d), whereas it rises in Figure 15 (d), and the deficit-to-national-income ratio is consistently lower in Figure 16 (e) than in Figure 15 (e). But this view neglects the fact that the upper thresholds for the public deficit ratio and the public debt ratio are considerably lower in Figure 16 than in Figure 15, mainly due to the higher safety discount factor in Figure 16 than in Figure 15. But in Figure 16 (d) the (declining) debt ratio always exceeds its upper threshold, whereas in Figure 15 (d) the (increasing) debt ratio always stays below its upper threshold. Hence, in Figure 15 (e) the initial primary deficit turns into a primary surplus, whereas in Figure 16 (e) the primary deficit lasts forever. Accordingly, after an initial increase, the present value of public debt falls to zero in Figure 15 (h), whereas it rises without any bound in Figure 16 (h).

If we lowered private households’ propensity to invest in government bonds in Figure 15 to the level that is assumed in Figure 16 the Maastricht criterion would also be fulfilled. This shows that, for a specific set of parameter constellations, the steady state debt ratio can be stable (which we termed medium term sustainable public debt) and, simultaneously, the government’s solvency constraint can be satisfied (which we termed long term sustainable public debt). This set of parameter constellations can be further divided into two subsets: for one the Maastricht criterion for the public debt ratio is fulfilled, for the other not. Thus, the Maastricht criterion for the public debt ratio cannot be justified by theoretical considerations of the sustainability of public debt.

The difference with respect to long term sustainability between Figure 15 and Figure 16 is based on the difference in the safety discount factor and hence the rate of interest which changes the sign of the primary-deficit-to-public–debt ratio, for which we get from eqs (95) and (148) along the SSL:

\[
\left(\frac{\hat{d}}{\hat{b}}\right)_{SSL} = s_{pr}^g (1 - \tau) - (1 - \alpha)(1 - \alpha)\left(1 - s_{pr}^g (1 - \tau)\right)\left(s_{pr}^g\right)^{-1} = s_{pr}^g (1 - \tau)\frac{b_{SSL}}{s_{pr}^g (1 - \tau)\left(1 - s_{pr}^g (1 - \tau)\right)}
\]  

(158).

The primary-deficit-to-public–debt ratio changes by almost the same amount in the opposite direction whenever the interest rate changes, e.g. due to a change in the safety discount factor. In Figure 17 we see that a change in the safety discount factor affects the rate of interest strongly and may even change the sign of the primary-deficit-to-public–debt ratio, whereas it leaves the growth rate of public debt and the phase curve of the debt ratio almost unchanged.
Figure 17: Impact of a change in the safety discount factor on the rate of interest, the primary-deficit-to-public–debt ratio, the growth rate of public debt, and the phase curve of the debt ratio

At least in principle, the upper thresholds for the public deficit ratio and the debt ratio according to eqs (127) and (128) could serve as alternatives to the arbitrary Maastricht criteria. As it might be difficult to observe the determinants of the upper thresholds, a primary budget surplus could be used instead. For this is a necessary and sufficient condition for the long term sustainability of a positive and medium term sustainable steady state public-debt-to-national-income ratio. But a fixed target of the primary surplus ratio may lead to other problems, as will be shown in section 6 below.

5. Austerity Measures Enforced

According to the Treaty on Stability Coordination and Governance (see European Commission 2013) for member states of the Euro Area the maximum budget deficit ratio is 1 %, and is even lower if the current public debt ratio exceeds 60 % (see European Commission 2013, especially pp 94 ff. for details concerning their definition and computation). In the case of Greece the fiscal target does not refer to the public deficit-to-GDP ratio, but to the primary surplus-to-GDP ratio instead (see European Commission 2012, p. 2 and European Commission 2015, pp 5ff). In the model presented above, the only policy parameter that can be adjusted by government accordingly is the tax rate. Hence, in the following we consider the cases where the parliament writes into law rule based tax rates which make sure that either a pre-established budget deficit-to-GDP ratio $B$ or a pre-established negative primary deficit-to-GDP ratio $\pi$ (which amounts to a positive primary surplus ratio) is continuously realized by setting the appropriate tax rate $\tau^H$ or $\tau^\pi$. According to eq. (19) there exists the following link between the two targets:

$$\frac{\dot{B}}{Y} = d + \eta b$$  \hspace{1cm} (159)

If the government targets the budget deficit ratio $\dot{B}/Y = \mu$ the primary deficit ratio $d$ changes with the ratio of government interest payments to national income $\eta b$. With a fixed target for the deficit ratio, an increase in the ratio of government interest payments to national income lowers the corresponding primary deficit ratio by the same amount. This stabilizes the dynamics of the debt ratio. Whereas, if the government targets the primary deficit ratio $d = \pi$ an increase in the ratio of government interest payments to national income increases the budget deficit ratio $\dot{B}/Y$ by the same amount. This destabilizes the dynamics of the debt ratio. This destabilizing effect is much stronger than in the case with a fixed tax rate, where an increase in the ratio of government interest payments to national income increases the
budget deficit ratio $B/Y$ just by the tiny fraction $s^g_{pr}(1-\tau)$. As the dynamics of the debt ratio are determined by the budget deficit ratio and the growth rate of national income (see eq. (132)), the government exerts much stronger control over debt dynamics if it targets the budget deficit ratio instead of a fixed primary surplus ratio. In the latter case the dynamics of the budget deficit ratio are determined by the dynamics of the ratio of government interest payments to national income, and hence to a large extent by ‘market forces’, i.e. investors, namely private households and their ‘demand rate of interest on government bonds’.

We first consider the case of the pre-established budget deficit-to-GDP ratio $\mu$. From eqs (15) and (52) we get:

$$\frac{\dot{B}}{Y} = \frac{s^g_{pr}}{Y} = s^g_{pr}(1-\tau^\mu)(1+r^\mu b^H) = \mu$$  \hspace{1cm} (160),

and hence for the tax rate $\tau^\mu$ with eq. (49):

$$\tau^\mu = 1 - \frac{\mu}{s^g_{pr}} \left(1+r^\mu b^H\right)^{-1} = 1 - \frac{\mu}{s^g_{pr}} \left(1+\lambda^{-1}(1-\sigma)(1-\alpha)\frac{k^\mu_{pr}}{k^\mu_{g}}\right)^{-1} ; 0 < \tau^\mu < 1$$  \hspace{1cm} (161).

The tax policy is rather straightforward in this case: Whenever the government’s interest payments-to-GDP ratio increases, the tax rate is increased in order to keep the ratio of households’ investment in public bonds to national income constant. Hence, $\tau^\mu$ varies with the ratio of public to private capital and is constant in the steady state (see Figure 19 (b)). By inserting eq. (161) into the system of differential equations (62) and (63) we obtain the following system of differential equations:

$$k^\mu_{pr} = \frac{s^f_{pr}}{s^g_{pr}} \mu \gamma \left(\frac{k^\mu_{pr}}{k^\mu_{g}}\right)^{1-\alpha-\beta} - (\delta + n)k^\mu_{pr}$$  \hspace{1cm} (162),

$$k^\mu_{g} = \lambda \mu \gamma \left(\frac{k^\mu_{pr}}{k^\mu_{g}}\right)^{1-\alpha-\beta} - (\delta + n)k^\mu_{g}$$  \hspace{1cm} (163).

Again, the steady state $(k^\mu_{pr}, k^\mu_{g})$ is derived by setting both eqs (162) and (163) equal to zero. By dividing the two resulting equations, we derive the steady state locus:

$$k^\mu_{g} = \frac{s^g_{pr}}{s^g_{pr}} k^\mu_{pr}$$  \hspace{1cm} (164),

which is identical to the one in the case of a fixed tax rate (see eq. (164) and eq. (64)) and is also stable. As the ratio of public to private capital is constant on the steady state locus, the tax rate is constant on the SSL, too. Again, on the SSL the rates of change of public and private capital and hence of the public and the private capital-to-labor ratios are equal. Hence, again, (i) a trajectory in a $k^\mu_{pr} - k^\mu_{g}$-phase diagram that starts on the SSL remains on the SSL; (ii) the SSL cannot be crossed by a trajectory. We obtain the steady state values of private and public capital per capita by inserting eq. (164) into eq. (162) and setting the differential equation equal to zero:
\[ \kappa_{pr}^* = \left( \frac{\gamma^{1-\alpha-\beta} (s_{pr}^{\alpha+\beta})^\mu}{(\delta + n)} \right)^{-1/\alpha} \]

and finally, again using eq. (164):

\[ \kappa_g^* = \left( \frac{\gamma^{1-\beta} (s_{pr}^\beta)^\mu}{(\delta + n)} \right)^{-1/\alpha} \]

For the steady state values to be positive, the deficit target ratio has to be positive. Figure 18 (a) shows a three dimensional Solow diagram of the differential equations of the private and the public capital-to-labor ratio, determining the steady state, Figure 18 (c) the corresponding debt ratio, its change in time and the steady state locus, Figure 18 (e) the increasing ratio of public interest payments to national income, the constant budget deficit ratio and the decreasing primary deficit ratio, and finally Figure 18 (g) the stream plot. Comparing Figure 18 (a), (c), (e), and (g) with the respective figures for a fixed tax (see Figure 3 (a), Figure 10 (a) and (c), and Figure 11 ) rate we notice only rather small differences between the two cases.

The rather small differences in the dynamics are mirrored by the derivatives of the budget deficit ratio with respect to the debt ratio in the two cases:

\[ \left( \frac{\dot{B}}{Y} \right)_{b}^\prime = s_{pr}^\prime (1-\tau) f_B (1+\epsilon_{B,b}) \geq 0 \]

(167),

\[ \left( \frac{\dot{B}}{Y} \right)_{b}^\mu = 0 \]

(168).

These differences correspond to differences in the derivatives of the primary deficit ratio with respect to the debt ratio:

\[ d_b^\prime = -(1-s_{pr}^\prime (1-\tau)) f_B (1+\epsilon_{B,b}) \leq 0 \]

(169),

\[ d_b^\mu = -f_B (1+\epsilon_{B,b}) \leq 0 \]

(170).

Obviously, the difference between the respective derivatives in the two cases is the smaller, the smaller the households’ propensity to invest in government bonds, the larger the tax rate and the closer \( \epsilon_{B,b} \) is to minus one, hence the closer the adjustment process is to the steady state locus or the steady state.

Furthermore, we can compute the growth rate of public debt along the steady state locus from eq. (148)

\[ \dot{g}_{SSL}^\mu = \frac{\mu}{b_{SSL}} \]

According to eq. (171), on the SSL the rate of change of public debt does not depend on the safety discount factor, whereas the rate of interest does according to eq. (91) which remains unaffected by the tax policy pursued in order to target a fixed budget deficit ratio. For the primary-deficit-to-public-debt ratio we can derive:
Figure 18: Budget deficit ratio target and primary surplus ratio target

\[
\begin{align*}
\frac{d}{dt} - \mu = \frac{\mu}{b} - \frac{(1-\sigma)(1-\alpha)}{\nu} \left( s_{\mu} f^2_{p} \right)^{-1} \frac{d}{d\nu}, \\
\frac{\partial}{\partial \nu} \mu = -1
\end{align*}
\]
which just changes by exactly the same amount in the opposite direction whenever the interest rate changes, e.g. due to a change in the safety discount factor. For the government to be solvent the primary deficit has to be negative. Hence, there exists an upper threshold that is not to be reached:

\[ \mu < \mu_{\text{Max}}^{\text{sol}} = (1 - \sigma)(1 - \alpha)(s_{pr}^{-1}) = b^* \]

for which the steady state rate of interest just equals the steady state growth rate of public debt. As the ratio of public interest payments to national income does not depend on the tax rate it is the same as in the cases with a fixed tax rate or a primary surplus ratio target.

By inserting the tax rate according to eq. (161) into eq. (30) we derive for the ratio of public consumption to national income:

\[ \left( \frac{g_s}{Y} \right)_{\mu} = 1 - \left( 1 - (1 - \lambda)s_{pr}^g \right) \frac{H}{s_{pr}} \frac{\partial (C_s/Y)}{\partial \mu} < 0 \]

Remarkably, the ratio of public consumption to national income is constant even outside the SSL, as it does not depend on the ratio of private to public capital. According to eq. (174) there exists a maximum budget deficit ratio target \( \mu_{\text{Max}}^{\text{liq}} \) that is mainly determined by the propensity to invest in government bonds:

\[ \mu_{\text{Max}}^{\text{liq}} = \frac{s_{pr}^g}{1 - (1 - \lambda)s_{pr}^g} ; \quad \frac{\partial \mu_{\text{Max}}^{\text{liq}}}{\partial \lambda} > 0 ; \quad \frac{\partial \mu_{\text{Max}}^{\text{liq}}}{\partial \mu} < 0 \]

and that is not to be exceeded if the liquidity constraint is to be satisfied. Whether the solvency or the liquidity threshold is larger largely depends on the size of the safety discount factor:

\[ \mu_{\text{Max}}^{\text{sol}} = \mu_{\text{Max}}^{\text{liq}} \Leftrightarrow 1 - \frac{s_{pr}^f}{(1 - \alpha)(1 - (1 - \lambda)s_{pr}^g)} \leq \sigma \]

For the government to be both liquid and solvent, and hence public debt sustainable, as the steady state is stable, the budget deficit ratio target has to fulfil the following condition:

\[ \mu^{\text{sus}} < \min \left( \mu_{\text{Max}}^{\text{liq}}, \mu_{\text{Max}}^{\text{sol}} \right) \]

For small levels of the safety discount factor the liquidity constraint is binding, for large ones the solvency constraint (see Figure 20 (a)). According to eqs (173), (175), and (177) an increase in the households’ propensity to invest in government bonds allows the government to target a higher budget deficit ratio without jeopardizing its solvency and liquidity. The more households shy away from government bonds the more difficult it becomes for the government to stay both solvent and liquid (see again Figure 20 (a)). As the efficiency condition (131) does not depend on the tax rate it also holds for a budget deficit ratio target. In Figure 20 (a) the light gray area shows combinations of the spread factor and the budget deficit ratio target for which the steady state debt ratio is long term sustainable and efficient.

According to eq. (161) the tax rate is a falling function of the budget deficit ratio target. The maximum budget deficit ratio target \( \mu_{\text{Max}}^{\text{liq}} \) corresponds to the minimum tax rate \( \tau_{\text{Min}}^{\text{liq}} \) according to eq. (121).

If a specific primary surplus-to-GDP ratio is established as a target we get from eq. (22):

\[ d = s_{pr}^g (1 - \tau^\pi) - \left( 1 - s_{pr}^g (1 - \tau^\pi) \right) b^\pi = \pi ; \quad \pi < 0 \]
The corresponding tax rate $\tau^\pi$ can be derived from eqs. (178) and (49) as:

$$\tau^\pi = 1 - \frac{\pi + \pi B^\pi}{s^g_{pr} (1 + B^\pi b^\pi)} = 1 - \frac{\pi + \lambda^{-1}(1-\sigma)(1-\alpha) \frac{\kappa^g_{pr} \pi}{\kappa^g_{pr}}}{s^g_{pr} (1 + \lambda^{-1}(1-\sigma)(1-\alpha) \frac{\kappa^g_{pr} \pi}{\kappa^g_{pr}})}, \quad \frac{\partial \tau^\pi}{\partial \pi} < 0, \quad \frac{\partial \tau^\pi}{\partial b^\pi} < 0 \tag{179}. $$

The tax policy to be pursued is rather different in this case: Whenever the government’s interest payments-to-GDP ratio increases and hence the primary surplus-to-GDP ratio increases for a given budget deficit-to-GDP ratio, the tax rate is reduced in order to keep the primary surplus-to-GDP ratio constant by increasing the ratio of households’ investment in public bonds to national income $s^g_{pr}$ and hence the budget deficit-to-GDP ratio. If we compare the partial derivatives of the tax rates with respect to the government’s interest payments-to-GDP ratio we note that they do not only differ with respect to the sign, but also with respect to their absolute value, as $1 - \pi \gg \mu$ (see Figure 19 (a)). Thus, the tax policy involved in case of a primary surplus ratio target is much more demanding than the one in case of a budget deficit ratio target. Again, $\tau^\pi$ varies with the ratio of public to private capital and is constant on the steady state locus (see Figure 19 (b)).
By inserting eq. (179) into the system of differential equations (62) and (63) we obtain the following system of differential equations:

\[
\begin{align*}
\dot{\pi}_{pr} &= \frac{\gamma S_{pr}}{\lambda S_{pr}} \left[ \pi \lambda \left( \kappa_{pr}^\pi \right)^{1-\alpha-\beta} + (1-\sigma)(1-\alpha) \left( \kappa_{pr}^g \right)^{1-\alpha-\beta} \right] - (\delta + n) \kappa_{pr}^\pi \\
\dot{\pi}_{g} &= \gamma \left[ \pi \lambda \left( \kappa_{pr}^\pi \right)^{1-\alpha-\beta} + (1-\sigma)(1-\alpha) \left( \kappa_{pr}^g \right)^{1-\alpha-\beta} \right] - (\delta + n) \kappa_{g}^\pi
\end{align*}
\]  

(180), (181).

Once more the steady state \( (\kappa_{pr}^*, \kappa_{g}^*) \) is derived by setting both eqs (180) and (181) equal to zero. By dividing the two resulting equations, we can see that, again, the following relationship holds in the steady state:

\[
\kappa_{g}^* = \frac{\lambda S_{pr}}{S_{pr}} \kappa_{pr}^*
\]  

(182).

Hence, again, on the SSL which can be shown to remain stable (see Figure 18 (h)) the tax rate is constant. And again the rates of change of public and private capital and hence of the public and the private capital-to-labor ratios are equal on the steady state locus. Therefore, again, (i) a trajectory in a \( \kappa_{pr} - \kappa_{g} \)-phase diagram that starts on the SSL remains on the SSL; (ii) the SSL cannot be crossed by a trajectory. We obtain the steady state values of private and public capital per capita by inserting eq. (182) into eq. (180):

\[
\begin{align*}
\kappa_{pr}^* &= \left[ \gamma \lambda^{1-\alpha-\beta} \left( \frac{s_{g}}{S_{pr}} \right)^{\alpha+\beta} \left( \pi + (1-\sigma)(1-\alpha) \left( \frac{s_{g}}{S_{pr}} \right)^{-1} \right) \right]^{-\frac{1}{\alpha}} \\
\kappa_{g}^* &= \left[ \gamma \lambda^{1-\beta} \left( \frac{s_{g}}{S_{pr}} \right)^{\beta} \left( \pi + (1-\sigma)(1-\alpha) \left( \frac{s_{g}}{S_{pr}} \right)^{-1} \right) \right]^{-\frac{1}{\alpha}}
\end{align*}
\]  

(183), (184).

For the steady state values to be positive, the primary surplus target ratio \( (-\pi) \) has to fulfill the following condition:

\[
-\pi < (-\pi)^{pos}_{Max} = (1-\sigma)(1-\alpha) \left( \frac{s_{g}}{S_{pr}} \right)^{-1} = \frac{r_{g}^* b^{\pi^*}}{r_{g}} \]  

(185),

which makes sure that the steady state budget deficit ratio is positive and is equivalent to the condition \( \mu > 0 \). Along the SSL the ratio of public consumption to national income does not change with the debt ratio, as it can be derived as:

\[46\] This condition is in contradiction to Sims' statement: "But if debt is real and the country finds itself unable to maintain primary surpluses above its predetermined real debt service commitment, it must default, even if in absolute terms it is running substantial primary surpluses." (Sims 2013 p. 568), but in line with inequality (100).
\[
\left( \frac{C_g}{Y} \right)^{\pi_{SSL}} = \left( \frac{C_g}{Y} \right)^{\pi^*} = 1 - \frac{1 - (1 - \lambda)s_{pr}^g}{s_{pr}^g} \left[ \pi + (1 - \sigma)(1 - \alpha) \left( s_{pr}^f \right)^{-1} \right];
\]
\[\frac{\partial \left( \frac{C_g}{Y} \right)^{\pi^*}}{\partial (1 - \sigma)} < 0; \quad \frac{\partial \left( \frac{C_g}{Y} \right)^{\pi^*}}{\partial (-\pi)} > 0\]

which is non-negative for:

\[-\pi \geq (-\pi)_{\min} = (1 - \sigma)(1 - \alpha) \left( s_{pr}^f \right)^{-1} - \frac{s_{pr}^g}{1 - (1 - \lambda)s_{pr}^g}\]

Hence, in order to simultaneously obtain positive steady state values of private and public capital and to satisfy the government’s liquidity constraint, the primary surplus target ratio has to fulfill the following condition:

\[(1 - \sigma)(1 - \alpha) \left( s_{pr}^f \right)^{-1} > (-\pi)_{\min} \geq (1 - \sigma)(1 - \alpha) \left( s_{pr}^f \right)^{-1} - \frac{s_{pr}^g}{1 - (1 - \lambda)s_{pr}^g}\]

The sign of the last term in (188) mainly depends on the size of the safety discount factor:

\[(-\pi)_{\min} \geq 0 \iff 1 - \frac{s_{pr}^f}{(1 - \alpha)(1 - (1 - \lambda)s_{pr}^g)} \leq \sigma\]

As a positive primary surplus ratio suffices to satisfy the government’s solvency constraint, fulfilling the liquidity constraint for a sufficiently small safety discount factor implies that the solvency constraint is satisfied as well (see Figure 20 (b)), and hence that the primary surplus ratio target is sustainable:

\[(1 - \sigma)(1 - \alpha) \left( s_{pr}^f \right)^{-1} > (-\pi)_{\min} \geq \max \left[ 0, (1 - \sigma)(1 - \alpha) \left( s_{pr}^f \right)^{-1} - \frac{s_{pr}^g}{1 - (1 - \lambda)s_{pr}^g} \right]\]

As a positive primary surplus ratio suffices to satisfy the government’s solvency constraint, fulfilling the liquidity constraint for a sufficiently small safety discount factor implies that the solvency constraint is satisfied as well (see Figure 20 (b)), and hence that the primary surplus ratio target is sustainable:

\[\begin{array}{ll}
\text{(a)} & \text{Budget deficit ratio target Exotaumue} \\
\text{(b)} & \text{Primary surplus ratio target Exotaupi}
\end{array}\]

\[\text{Figure 20: Combinations of budget deficit / primary surplus ratio target and spread factor for which steady state public debt is sustainable and efficient (} s_{pr}^f = 0.1; s_{pr}^g = 0.04; \gamma = t; \alpha = 0.7; \beta = 0.2; \delta = 0.02; n = 0 \} \]

Accordingly, sustainability of a positive steady state with respect to both solvency and liquidity requires the primary deficit ratio target to be negative and to lie within the range between the blue and the red line in Figure 20 (b). This range crucially depends on the households’ propensity to invest in government bonds. As the efficiency condition (131) does not depend on the tax rate it also holds for a primary
surplus ratio target. In Figure 20 (b) the light gray area shows combinations of the spread factor and the primary surplus ratio target for which steady state debt ratio is positive, long term sustainable and efficient. For the targets of the deficit ratio and the primary deficit ratio to be mutually compatible, according to eq. (95) the primary ratio target must equal the deficit ratio target minus the steady state ratio of public interest payments to national income:

\[ \pi^* = \mu - (1 - \sigma)(1 - \alpha)(s_{pr}^{-1})^{-1} = \hat{B} \varphi \pi - (1 - \sigma)(1 - \alpha)(s_{pr}^{-1})^{-1} \]

(191).

By inserting eq. (191) into eqs (183) and (184) we observe that the steady state values of the private and public capital-to-labor ratios are identical for mutually compatible targets of the public deficit ratio and the primary deficit ratio. From eq. (191) we see that there is a linear relationship between the two targets with a slope of one. A reduction of the target rate of the budget deficit ratio by one percent reduces the corresponding target rate of the primary deficit ratio by one percent, too.47 This is the reason why the light gray areas in Figure 20 (a) and Figure 20 (b) are of equal size. By inserting eq. (90) into eq. (161) or eq. (179) we derive the steady state tax rate for both (mutually compatible) targets:

\[ \tau^* = \mu^* = 1 - \frac{\mu}{s_{pr}^{-1} + (1 - \sigma)(1 - \alpha)(s_{pr}^{-1})^{-1}} = \tau^* = 1 - \frac{\pi + (1 - \sigma)(1 - \alpha)(s_{pr}^{-1})^{-1}}{s_{pr}^{-1} + (1 - \sigma)(1 - \alpha)(s_{pr}^{-1})^{-1}} \]

(192).

The smaller the target rate of the budget deficit ratio and the higher the propensity to invest in government bonds, the higher is the required steady state tax rate. (See also Figure 19.) If eqs (192) are fulfilled the steady state values of the various variables are identical in the three cases as are their dependency on the growth rate \((\delta + n)\). Hence, Figure 6 is also valid with a budget deficit and a surplus ratio target.

If we compute the Jacobian matrices for a budget deficit and primary surplus ratio target we find that trace, determinant and discriminant are the same as in the case with a fixed tax rate. Hence, the steady states are locally asymptotically stable nodes, too. Table 3 summarizes the conditions for liquidity, solvency, stability, long term sustainability, and efficiency of public debt with a budget deficit and a primary surplus ratio target.

For mutually compatible steady state values of the tax rate, the budget deficit ratio target and the primary surplus target the steady state loci coincide. As the SSL is identical in all three cases, on the SSL the dynamics of the debt ratio are the same with a budget deficit ratio or a primary surplus ratio target as with a fixed tax rate. Accordingly, the corresponding differential equations for the debt ratio along the SSL are also identical for compatible values of the tax rate, the budget deficit ratio target and the primary surplus ratio target. Hence, for the case of a primary surplus ratio target we can write according to eqs (90), (191), and (151):

47 Hence, as long as the propensity to invest in government bonds and the safety discount factor are exogenous, one can either state a target rate for the budget deficit ratio, e.g. according to the Treaty on Stability Coordination and Governance (see European Commission 2013), or a target rate for the primary surplus ratio, as in the Second Economic Adjustment Programme for Greece (see European Commission 2012, p. 2) or in the Memorandum of Understanding between the European Commission Acting on Behalf of the European Stability Mechanism and the Hellenic Republic and the Bank of Greece on 19 August 2015 (European Commission 2015, pp 5ff).
\[ \dot{b}^\pi_{SSL} = \alpha \left[ \pi + (1 - \sigma)(1 - \alpha) \left( \frac{f_g}{s_{pr}} \right)^{-1} \right] - \alpha (\delta + n)b^\pi_{SSL} \]  
(193).

and for the case of budget deficit ratio target:

\[ \dot{b}^\mu_{SSL} = \alpha \mu - \alpha (\delta + n)b^\mu_{SSL} \]  
(194).

<table>
<thead>
<tr>
<th>Liquidity</th>
<th>Stability</th>
<th>Solvency</th>
<th>Long Term Sustainability</th>
<th>Efficiency</th>
</tr>
</thead>
</table>
| **Endogenous rate of interest: budget deficit ratio target**<br>\[ \mu < \mu_{\text{Max}} = \frac{s_{pr}^g}{1 - (1 - \lambda)s_{pr}^g} \]  
\[ \alpha > 0 \cup (\delta + n) > 0 \]  
\[ \mu < \mu_{\text{Max}} = \frac{s_{pr}^g}{s_{pr}^f (1 - \sigma)(1 - \alpha)} \]  
\[ 0 < \mu < \min \left\{ \frac{s_{pr}^g}{1 - (1 - \lambda)s_{pr}^g}, \frac{s_{pr}^g}{s_{pr}^f (1 - \sigma)(1 - \alpha)} \right\} \]  
\[ 1 - \sigma \leq (1 - \alpha)_{\text{Max}} = \frac{(1 - \alpha - \beta) s_{pr}^f}{s_{pr}^f (1 - \alpha) (1 - \alpha) s_{pr}^g} \]  
\[ (\delta + n) > 0 \]  
\[ \text{Endogenous rate of interest: primary surplus ratio target} \]  
\[ \frac{s_{pr}^g (1 - \sigma)(1 - \alpha)}{s_{pr}^f} > \left( \frac{\alpha > 0 \cup (\delta + n) > 0}{-\pi} > 0 \right) \]  
\[ \frac{s_{pr}^g (1 - \sigma)(1 - \alpha)}{s_{pr}^f} > \left( \frac{\alpha > 0 \cup (\delta + n) > 0}{-\pi} > 0 \right) \]  
\[ \frac{s_{pr}^g (1 - \sigma)(1 - \alpha)}{s_{pr}^f} > \left( \frac{\alpha > 0 \cup (\delta + n) > 0}{-\pi} > 0 \right) \]  
\[ \frac{s_{pr}^g (1 - \sigma)(1 - \alpha)}{s_{pr}^f} > \left( \frac{\alpha > 0 \cup (\delta + n) > 0}{-\pi} > 0 \right) \]  
\[ 0 < \mu < \max \left\{ \frac{s_{pr}^g}{1 - (1 - \lambda)s_{pr}^g}, \frac{s_{pr}^g}{s_{pr}^f (1 - \sigma)(1 - \alpha)} \right\} \]  
\[ 1 - \sigma \leq (1 - \alpha)_{\text{Max}} = \frac{(1 - \alpha - \beta) s_{pr}^f}{s_{pr}^f (1 - \alpha) (1 - \alpha) s_{pr}^g} \]  
\[ \alpha > 0 \cup (\delta + n) > 0 \]  

Table 3: Endogenous rate of interest: austerity measures enforced - liquidity, solvency, stability, long term sustainability, and efficiency of public debt

Furthermore we can compute the rate of change of public debt along the SSL according to eqs. (171) and (191):

\[ \dot{b}^\pi_{SSL} = \frac{\pi + (1 - \sigma)(1 - \alpha)\left( \frac{f_g}{s_{pr}} \right)^{-1}}{b^\pi_{SSL} + f^B_{SSL}} = \frac{\pi}{b^\pi_{SSL} + f^B_{SSL}} \frac{\partial b^\pi_{SSL}}{\partial \sigma} < 0 \]  
(195).

Comparing eqs (171) and (195) shows, that in case of a primary surplus ratio target, the rate of change of public debt is affected by a change in the safety discount factor via the impact on the rate of interest, whereas this is not the case with a budget deficit ratio target. A change in the rate of interest changes the growth rate of public debt by exactly the same amount when a primary surplus ratio is targeted. Obviously, for the primary-deficit-to-public–debt ratio holds

\[ \left( \frac{\pi}{b^\pi_{SSL}} \right) = \frac{\partial b^\pi_{SSL}}{\partial \sigma} \]  
(196),

which is completely independent from the safety discount factor and hence a change in the rate of interest. For the primary surplus ratio target Figure 18 (b) shows the three dimensional Solow diagram of the
differential equations of the private and the public capital-to-labor ratio determining the steady state, Figure 18 (d) the corresponding debt ratio, its change in time and the steady state locus, Figure 18 (f) the corresponding increasing ratio of public interest payments to national income, the increasing budget deficit ratio and the constant primary deficit ratio, and finally Figure 18 (h) the stream plot. Comparing Figure 18 (b), (d), (f), and (h) with the respective figures for a fixed tax rate and a fixed budget deficit ratio we notice important differences between the first two cases and the last one.

The rather large difference in the dynamics far away from the SSL is mirrored by large differences in the derivatives of the budget deficit ratio with respect to the debt ratio in the three cases. For the primary surplus ratio target we get:

$$\frac{d\pi'}{\pi} = 0 \quad (198)$$

Obviously, the difference between the respective derivatives in the cases with a fixed tax rate and a budget deficit ratio target on the one side and with a primary deficit ratio target is the smaller, the closer $\varepsilon_{\delta, b}$ is to minus one, hence the closer the adjustment process is to the steady state locus or the steady state.

The differences in the stream plots for the three cases show up in the partial derivatives of the differential equations. Whereas the partial derivative of the differential equation for the public-capital-to-labor
ratio with respect to the public-capital-to-labor ratio is negative with a fixed tax rate and a budget deficit ratio target, it is positive with a primary deficit ratio target. Hence, in the latter case, for the trace to be unchanged, the partial derivative of the differential equation for the private capital-to-labor ratio with respect to the private capital-to-labor ratio is more negative than with a fixed tax rate and a deficit ratio target. This implies that the trajectories of the private and the public capital-to-labor ratios whose starting point is not located in the vicinity of the SSL do not directly approach the steady state, but the $\kappa_{pr}^{\pi} = 0$-isocline instead.

At the $\kappa_{pr}^{\pi} = 0$-isocline the trajectories sharply change direction with respect to the private capital-to-labor ratio and enter into the corridor between the SSL and the $\kappa_{pr}^{\pi} = 0$-isocline, within which they then directly approach the steady state. Only if the starting point is situated in the vicinity of the SSL, the steady state is approached directly, i.e. without a change in direction of the trajectory. Hence, for $\kappa_{pr}^{\pi} < \kappa_{pr}^{\pi*}$ ($\kappa_{pr}^{\pi} > \kappa_{pr}^{\pi*}$) the value of the public-capital-to-labor ratio on the $\kappa_{pr}^{\pi} = 0$-isocline is the lower (upper) limit of the corridor around the SSL within which the trajectories (finally) approach the steady state without further change in direction.

As the steady state is a stable node, not a stable focus, the trajectories change direction with respect to both the private and the public capital-to-labor ratio at most once. With a primary surplus ratio target the $\kappa_{pr} = 0$-isocline plays a more important role for the dynamic behavior outside the steady state and the steady state locus than with a budget deficit ratio target or with a fixed tax rate, as the stream plots show.

Whereas the dynamic behavior is identical along the SSL in the three cases, it is different along the $\kappa_{pr} = 0$-isocline, especially among the cases with a fixed tax rate and budget deficit ratio target on the one side (see Figure 14 (a) and (b) as well as Figure 21 (a) and (b)) and the primary surplus ratio target on the other (see Figure 21 (c) and (d)). Along the $\kappa_{pr} = 0$-isocline, the budget deficit ratio remains constant with a budget deficit ratio target or almost constant with a fixed tax rate if the debt ratio increases, whereas the budget deficit ratio increases with a primary surplus ratio target.

Accordingly, the (negative) slope of the differential equation of the debt ratio in the steady state is larger with a primary surplus ratio target. In this sense, the steady state is ‘less stable’, i.e. the adjustment process to the steady state takes more time (see Figure 22 (c) and (d)). Depending on the starting point (which is identical in Figure 22 (a) and (b)) the trajectory may even lead away from the steady state with a primary surplus ratio target. This is not the case with a budget deficit ratio target. Hence, during the adjustment process, with a primary surplus ratio target the debt ratio starting from a level above equilibrium may further increase before it decreases. Again, this is not the case with a budget deficit ratio target or a fixed tax rate.

By comparing eqs (167), (168), and (197), we obtain for the impact of a shock to the debt ratio on the budget deficit ratio:

$$ \frac{\dot{B}^{\mu}}{\dot{Y}^{\mu}} \leq \frac{\dot{B}^{\pi}}{\dot{Y}^{\pi}} \leq \frac{\dot{B}^{\pi}}{\dot{Y}^{\pi}} $$

(199)

and by comparing eqs (169), (170), and (198) for the impact of a shock to the debt ratio on the primary deficit ratio

$$ d_{b}^{\mu} \leq d_{b}^{\pi} \leq d_{b}^{\pi} $$

(200)
Whereas the impact of an increase in government’s interest payments resulting from a positive shock to public debt on the budget deficit is largely compensated by a decrease of the primary deficit in the case of a fixed tax rate, it is completely offset by a decrease in the primary deficit if a fixed budget deficit ratio is targeted, but not at all if a fixed primary deficit ratio is targeted.

\[(a) \text{ Budget deficit ratio target ( } \mu = 0.014 \text{ )} \quad (b) \text{ Primary surplus ratio target ( } \pi = -0.01 \text{ )} \]

\[(c) \text{ Budget deficit ratio target ( } \mu = 0.014 \text{ )} \quad (d) \text{ Primary surplus ratio target ( } \pi = -0.01 \text{ )} \]

For starting points outside the steady state locus, the steady state is approached in the most direct way in case of a budget deficit target, and in the least direct way in case of a primary surplus target, as the former avoids the destabilizing impact of the rate of interest throughout, whereas the latter adds a destabilizing change in the tax rate which is reduced if the debt ratio increases in order to keep the primary surplus ratio constant and thus prevents the stabilizing decline of the primary deficit ratio in case of an increase in the debt ratio. As the ratio of public consumption to national income is constant with a budget deficit ratio target even outside the SSL (see Figure 19 (c) and (d)), illiquidity is no additional concern outside the SSL if sustainability condition (177) is satisfied in the steady state. This is quite different with a primary surplus target where the ratio of public consumption to national income varies sharply outside the SSL (see again Figure 19 (c) and (d) and also Figure 22 (a) and (b)). By implicit differentiation of eq. (160) we get for a constant budget deficit ratio:

\[
\frac{dt^H}{db^H} = \frac{(1 - t^H)I^H}{1 + \varepsilon^H b^H} \geq 0 \text{ for } 0 > \varepsilon^H b^H, \frac{t^H}{b^H} \geq -1
\]

and by implicit differentiation of eq. (22) we get for a constant primary surplus ratio
If we compare eq. (201) to eq. (202) we see that in case of a budget deficit ratio target an increase in the debt ratio leads to an increase in the tax rate which dampens the demand for newly issued government bonds and hence implies a negative feedback on the dynamics of the debt ratio, whereas in case of a primary surplus ratio target an increase in the debt ratio reduces the tax rate which increases the demand for newly issued government bonds and hence implies a positive feedback on the dynamics of the debt ratio. But these feedbacks vanish, as the steady state is approached and both $\varepsilon_{bB} \mu B$ and $\varepsilon_{bg} b^\mu \pi$ tend to minus one.

Let us finally consider the respective differential equations for the debt-to-national income ratio. For the budget deficit target we get:

$$\dot{b}^\mu = \mu - \dot{\gamma}^\mu b^\mu \quad (203).$$

With $\dot{\gamma}^\mu = \dot{\gamma}^\mu = \delta + n$ we get the linear approximation:

$$\dot{b}^{ul} = \mu - (\delta + n) b^{ul} \quad (204).$$

If we compare this linear approximation to the differential equation for the debt-to-national income ratio along the steady state locus we get:

$$\dot{b}^{ul} = (1/\alpha) b^{SSL} \quad (205).$$

From eq. (203) follows the steady state debt ratio with a budget deficit ratio target:

$$b^\mu = \frac{\mu}{\delta + n}; \frac{\partial b^\mu}{\partial \delta} < 0 \quad (206),$$

whereas with a primary deficit ratio target according to eq. (193):

$$b^\pi = \frac{\pi + (1 - \sigma)(1 - \alpha)}{\delta + n} \left( s_{pr}^B \right)^{-1}; \frac{\partial b^\pi}{\partial \delta} < 0, \frac{\partial b^\pi}{\partial \pi} > 0 \quad (207).$$

According to eqs (206) and (207) the steady state debt ratio rises with an increase in the deficit ratio target and falls with a decrease in the primary deficit ratio target. For the primary deficit ratio target we can derive:

$$\dot{b}^\pi = \pi - (\dot{\gamma}^\pi - r_B) b^\pi \quad (208).$$

If we compare eq. (208) to eq. (140) we observe a strong similarity of the two. Hence, G2’s approach may be interpreted as a public debt policy with a fixed primary debt ratio target. As a primary surplus ratio target implies $\dot{\gamma}^\pi < r_B$, G2 might expect the steady state to be unstable. But from the stability analysis above we know that the steady state is a locally stable node, independent from the primary budget ratio target being positive or negative. Hence, G2’s approximation approach also leads to misleading results with respect to the stability of the steady state in case of a primary surplus ratio target with an endogenously determined rate of interest according to eq. (41). Instead, according to Figure 23, this is not the case if either the steady state locus or the $\kappa_{pr}^\pi = 0$-isocline is used for approximation.
Finally, we compare the impact of a change in the safety discount factor on the rate of interest, the primary-deficit-to-public–debt ratio, the growth rate of public debt, and the phase curve of the debt ratio for a budget deficit ratio target and a primary surplus ratio target. From Figure 24 (a) we see that with a budget deficit ratio target a change in the safety discount factor changes the rate of interest and the sign of the primary-deficit-to-public–debt ratio, whereas it leaves the growth rate of public debt and the phase curve of the debt ratio completely unchanged (see eqs (91), (171), (172), and (194)). Quite differently, from Figure 24 (b) we see that with a primary surplus ratio target a change in the safety discount factor changes the rate of interest and the growth rate of public debt by the same amount, shifts the phase
curve of the debt ratio by the change of the rate of interest multiplied by the production elasticity of labor and leaves the primary-deficit-to-public–debt ratio completely unchanged (see eqs (91), (195), (196), and (193)).

(a) Budget deficit ratio target (μ = 0.014)
(b) Primary surplus ratio target (π = −0.01)

Figure 24: Budget deficit ratio target and primary surplus ratio target compared: impact of a change in the safety discount factor on the rate of interest, the primary-deficit-to-public–debt ratio, the growth rate of public debt, and the phase curve of the debt ratio

\( s_p = 0.02; s_g = 0.1; \gamma = 1; \alpha = 0.7; \beta = 0.2; n = 0; \delta = 0.02; \lambda = 1 \)

Summing up, we found that, compared with a fixed tax rate, enforcing austerity measures by targeting either the budget deficit ratio or the primary surplus ratio does not fundamentally change the steady state behavior of public debt with respect to liquidity, stability, solvency, and efficiency, as long as the propensity to invest in government bonds is exogenous and constant as well as the rate of interest on government bonds is tied to the rate of return on private capital via an exogenous spread factor. The latter implies that, at least, along the SSL, which is stable, the rate of interest on government falls when the debt ratio increases.

6. Fixed Rate of Interest on Government Bonds

In the case of the European programme countries the rate of interest on government bonds is largely determined by political institutions like the European Commission, the European Central Bank, and the International Monetary Fund, not by market forces. Hence, in the following we treat the rate of interest on government bonds as exogenously given:

\[ r_B = r_B \] (209)

As the propensities to invest remain exogenously given, the steady state locus and its stability also remain unchanged in the three cases, namely with a fixed tax rate, a budget deficit ratio target and a primary surplus ratio target. The SSL remains a straight line with identical percentage rates of change of the private and the public capital-to-labor ratios. Hence, again, if the initial combination of the private and the public capital-to-labor ratio is located on the SSL the development remains on the SSL in the phase diagram. But now the ratio of the government’s interest payments to national income varies along the SSL due to variations of the debt ratio, which are no longer compensated by inverse variations of the rate of interest on government bonds. Hence, the first equality in eqs (93) is no longer valid, only the second and third one. According to eq. (57) we get for the shadow rate of return on the public deficit along the SSL:

\[ \overline{s}r_B^{\text{SSL}} = (1 - \alpha - \beta)(\overline{b}^{\text{SSL}})^{-1} \] (210),

58
where the upper bar denotes the variables in case of an exogenous rate of interest on government bonds. Accordingly, the shadow rate of return on the public deficit exceeds the exogenous rate of interest for

$$\bar{b}^{\text{SSL}} < \bar{b}^{\text{eff}} = \frac{1-\alpha-\beta}{\tau_B}$$  \hspace{1cm} (211),

where $\bar{b}^{\text{eff}}$ denotes the maximum debt ratio for which the efficiency condition of public budget deficits is satisfied. According to eq. (211), $\bar{b}^{\text{eff}}$ is the larger, the larger public capital’s production elasticity $(1-\alpha-\beta)$ and the smaller the exogenously given rate of interest on government debt. From eqs (23) and (209) we can derive for the budget deficit ratio along the SSL:

$$\left(\frac{\dot{B}}{Y}\right)^{\text{SSL}} = \frac{s_{pr}^g(1-\tau)}{b^{\text{SSL}}}(1+\tau_B\bar{b}^{\text{SSL}})$$  \hspace{1cm} (212)

in the case of a fixed tax rate. Now, the budget deficit ratio is increasing with the debt ratio along the SSL, whereas it is unchanged in the case of an endogenous interest rate with a fixed tax rate considered above (see eq. (96)). Yet, the rate of change of public debt along the SSL declines with the debt ratio in case of a fixed interest rate, too:

$$\frac{\dot{b}}{b^{\text{SSL}}} = \frac{s_{pr}^g(1-\tau)}{b^{\text{SSL}}} + \frac{s_{pr}^g(1-\tau)\tau_B}{(1-s_{pr}^g(1-\tau))\tau_B}$$  \hspace{1cm} (213).

For the primary-deficit-to-debt ratio we get along the SSL:

$$\left(\frac{\dot{d}}{d}\right)^{\text{SSL}} = \frac{s_{pr}^g(1-\tau)}{b^{\text{SSL}}} - \left(1-s_{pr}^g(1-\tau)\right)\tau_B$$  \hspace{1cm} (214)

whose sign depends on

$$\dot{b}^{\text{SSL}} \geq \bar{b}(\bar{d} = 0) = \frac{s_{pr}^g(1-\tau)}{\left(1-s_{pr}^g(1-\tau)\right)\tau_B}$$  \hspace{1cm} (215).

Hence, for a fixed interest rate a primary surplus requires a sufficiently large debt ratio. The smaller the rate of interest, the larger the debt ratio has to be. According to eq. (151) we can derive the following differential equation for the debt ratio along the SSL:

$$\frac{\dot{b}}{b^{\text{SSL}}} = \alpha s_{pr}^g(1-\tau) - \alpha \left(\delta + n - s_{pr}^g(1-\tau)\tau_B\right)b^{\text{SSL}}$$  \hspace{1cm} (216)

with the steady state value of the debt ratio:

$$\bar{b}^* = \frac{s_{pr}^g(1-\tau)}{(\delta + n) - \tau_B s_{pr}^g(1-\tau)}; \quad \frac{\partial \bar{b}^*}{\partial \tau_B} < 0; \quad \frac{\partial^2 \bar{b}^*}{\partial \tau_B^2} > 0$$  \hspace{1cm} (217),

which increases if the rate of interest increases. It is positive and asymptotically stable, as long as the maximum threshold of the interest rate $\bar{\tau}_B^{\text{sta}}$ is not reached

$$\bar{\tau}_B < \bar{\tau}_B^{\text{sta}} = \frac{(\delta + n)}{s_{pr}^g(1-\tau)}$$  \hspace{1cm} (218).
For a given interest rate follows the minimum threshold of the tax rate $\tau_{\text{Min}}^{\text{sta}}$ that has to be exceeded in order to ensure stability of the steady state:

$$\tau > \tau_{\text{Min}}^{\text{sta}} = 1 - \frac{\delta + n}{\gamma_{\text{pr}} B} - \frac{\sigma_{\tau_{\text{Min}}}^{\text{sta}}}{\sigma_{\tau B}^{\text{sta}}} > 0$$ (219).

This minimum threshold is negative for a sufficiently small interest rate. As in case of a variable interest rate the steady state remains stable even if the rate of interest exceeds the steady state growth rate, as long as it does so by a sufficiently small amount. Yet, if we compare the slopes of the differential equations for the debt ratio in the two cases (see eqs (151) and (216)) we observe the slope to be larger in absolute terms with a variable rate of interest than with a fixed one. In this sense the former is 'more stable' than the latter. This is in line with the fact that, according to eqs (92) and (218), the maximum threshold of the interest rate $\tau_{\text{Max}}^{\text{sta}}$ exceeds the endogenous steady state rate of interest in case of a fixed tax rate. Only if the interest rate is determined exogenously can it be sufficiently excessively large in order to lead to instability.

By inserting eq. (217) into eq. (214) we get for the steady state primary surplus ratio:

$$-d^* = s_{d_{\text{pr}}}^B (1 - \tau) (\tau_B - \delta - n); \quad \frac{\partial (-d^*)}{\partial (\delta + n)} < 0; \quad \frac{\partial (-d^*)}{\partial \tau B} > 0 \quad \forall \quad \tau_B > (\delta + n) > \tau_B s_{d_{\text{pr}}}^B (1 - \tau)$$ (220),

and hence for the primary-surplus-to-debt ratio:

$$-\frac{d^*}{b^*} = \tau_B - \delta - n; \quad \frac{\partial (-d^*/b^*)}{\partial (\delta + n)} < 0; \quad \frac{\partial (-d^*/b^*)}{\partial \tau B} > 0$$ (221),

and the budget deficit ratio:

$$\left(\frac{B}{Y}\right)^* = -d^* + \tau_B b^* = (\delta + n) b^* = \frac{(\delta + n) s_{d_{\text{pr}}}^B (1 - \tau)}{(\delta + n) - \tau_B s_{d_{\text{pr}}}^B (1 - \tau)}; \quad \frac{\partial (B/Y)^*}{\partial (\delta + n)} < 0; \quad \frac{\partial (B/Y)^*}{\partial \tau B} > 0$$ (222).

Other than in case of a variable interest rate, an increase in the growth rate changes the sign of both the difference between the steady state growth rate and the interest rate and of the primary deficit (see Figure 25 (a)). But, in both cases in a growing economy ($\delta + n > 0$) a constant and long term sustainable real steady state debt ratio requires the steady state budget deficit as well as the (positive) primary surplus to be smaller than the government's interest payments, a condition that is fulfilled in Figure 25 (a) for $\delta + n < \tau_B$. As long as the growth rate is smaller than the interest rate, a primary surplus is generated in the steady state. Hence, for the government to be solvent in the steady state, from (215) and (217) follows:

$$b^* > b(d = 0)$$ (223),

i.e. the steady state debt ratio has to be sufficiently large (see Figure 25 (b)).\(^{48}\) For $\delta + n < \tau_B$, the larger and hence the closer the growth rate is to the interest rate, the lower is the steady state primary surplus ratio, i.e. the primary surplus ratio that is required in order to render the steady state debt ratio sustainable. On the steady state path the primary surpluses grow at the steady state growth rate, too. Hence,

\(^{48}\) In a somewhat similar way, in Figure 15 the government becomes solvent with a variable interest rate, once the debt ratio becomes large enough.
the faster primary surpluses grow in the future, the lower their required current level, even relative to public debt according to eq. (221) - a sort of substitution over time of primary surplus requirements.49

If we compare the differential equation (216) with a fixed rate of interest to the differential equation (151) with a variable rate of interest we see that the slope is more negative in the latter case. Hence, fixing the rate of interest destabilizes the steady state debt ratio somewhat without making it unstable for many combinations of parameter values.50 For the steady state debt ratio with a fixed rate of interest to be both sustainable and efficient, the following inequalities must hold according to eqs (129) and (211):

\[ s_{pr}^B - \alpha \beta + \delta \tau < 0 \]

For \( \lambda > 0 \) the second inequality in (224) implies inequality (218). Under condition (218) the steady state is stable, as the differential equation for the debt ratio along the SSL (216) is negatively sloped. This condition is necessary for the determinant of the Jacobian matrix of the corresponding system of differential equations for the private and public capital-to-labor ratios to be positive, and sufficient for its trace to be negative. This raises the question of how probable it is that condition (218) is not fulfilled. Obviously, it is much more likely that the rate of interest on government bonds exceeds the steady state growth rate than its multiple \( \lambda \). But in a stagnant economy with a steady state growth rate close to zero it cannot be excluded. In case that condition (218) is not fulfilled the steady state is a saddle point with negative steady state values of private and public capital, and of public debt. Hence, in the latter case a positive debt ratio tends to grow without limit.

With a low interest rate it is certainly easier to serve debt. Hence, the proposition is somewhat counter-intuitive that a sufficiently high rate of interest is necessary in order to satisfy the solvency constraint, as ceteris paribus an increase in the rate of interest on government bonds increases the rate of change of public debt. On the other hand it is intuitive that a sufficiently large primary surplus is a necessary condition for the government’s ability to pay back its debt, hence for its solvency. In order to reconcile the two propositions the primary-deficit-to-public-debt ratio has to fall with an increase in the interest rate. Hence, according to eq. (21) the following condition has to be fulfilled:

\[ B = \frac{(1 - \alpha - \beta)(\delta + n)}{s_{pr}^B (1 - \tau)(2 - \alpha - \beta)} \]

49 See in this context fn 30 above, too. This assumes that government bonds are either consols or can easily be rolled over after maturity. In any case, investors in government bonds are supposed to be patient.

50 This holds in comparison to a situation where the interest rate falls with the debt ratio, at least along the SSL, as in the model presented above.
\[
\frac{\partial (d/b)}{\partial \bar{r}_B} = \frac{\partial \bar{B}}{\partial \bar{r}_B} - 1 < 0 \tag{225},
\]

i.e. the growth rate of public debt has to increase less than the interest rate. With a fixed tax rate this condition is fulfilled along the SSL according to eq. (214):

\[
\frac{\partial (d/b)}{\partial \bar{r}_B}^{SSL} = s_{pr}^g (1 - \bar{r}) - 1 < 0 \tag{226}.
\]

Figure 26: Impact of a change in the rate of interest on the growth rate of public debt, the phase curve of the debt ratio, and the primary-deficit-to-public–debt ratio

From eq. (226) and Figure 26 we see that an increase in the rate of interest reduces the primary-deficit-to-public–debt ratio at any debt ratio, a result that is in line with the empirical findings of Debrun and Kinda 2013, p. 4. The impact of a change in the interest rate on the primary-deficit-to-public–debt ratio is large due to its impact on the growth rate of public debt and hence the dynamics of public debt being small. As long as the rate of interest is smaller than the steady state growth rate of national income an increase in the rate of interest above this level is to be welcomed as the ‘cost’ due to the increase in the debt ratio is of lesser importance than the ‘benefit’ from the improvement of the primary balance which finally leads to government’s solvency. This ‘interest rate discipline’ forces the government via the liquidity constraint to run a primary surplus if the rate of interest exceeds the steady state growth rate of public debt. Furthermore, for a given tax rate, a given public-investment-to-public-deficit ratio, and a given households’ propensity to invest in government bonds, the liquidity constraint forces the government to reduce its consumption expenditures when its interest payments rise (see eq. (30)). For a given rate of interest we can compute from eq. (30) the minimum tax rate ensuring the liquidity constraint to be fulfilled in the steady state:

\[
\bar{r}_{Min}^{liq} = 1 - \frac{1}{1 - (1 - \lambda) s_{pr}^g (1 + \bar{B}^{p} \bar{r})} \cdot \frac{\partial \bar{r}_{Min}^{liq}}{\partial \bar{r}_B} > 0 \tag{227}.
\]

According to eq. (217) it increases with the rate of interest. As the ratio of public interest payments to national income increases with the debt ratio along the SSL the ratio of public consumption to national income declines according to eq. (30). It vanishes for
\[
\frac{b^\text{liq}_{\text{Max}}}{1} = \frac{1}{\tau_B} \left[ \frac{1}{1 - (1 - \lambda)s^g_{\text{pr}}(1 - \tau)} - 1 \right]
\]

(228),

which is very large for reasonable parameter values.

From eqs (219) and (227) we can compute the difference between the minimum tax rate ensuring the liquidity constraint to be fulfilled in the steady state and the minimum threshold of the tax rate that has to be exceeded in order to ensure stability of the steady state:

\[
\tau^\text{liq}_{\text{Min}} - \tau^\text{sta}_{\text{Min}} = \frac{(\delta + n)^2 (1 - (1 - \lambda)s^g_{\text{pr}})}{s^g_{\text{pr}} \tau_B \left[ \delta + n \left( 1 - (1 - \lambda)s^g_{\text{pr}} \right) \right] + s^g_{\text{pr}} (\tau_B - \delta(1 - \lambda))} > 0 \quad \forall \delta + n \cup 0 \leq \lambda \leq 1
\]

(229)

which is positive for a positive steady state growth rate. Hence, if the liquidity constraint is satisfied the steady state is stable.

(a) Ratios of public consumption to national income depending on private and public capital-to-labor ratios (\( T_B = 0.035; \mu = 0.015 \))

(b) Combinations of interest rate and tax rate for which steady state public debt is long term sustainable and efficient

The light gray area in Figure 27 (b) shows the combinations of the tax rate and the interest rate for which the steady state public debt is long term sustainable and efficient. In Figure 27 (b) \( \tau^\text{sta}_{\text{Min}} \) is not shown as it is negative. If we compare Figure 27 (b) with Figure 9 we see that in case of an exogenous interest rate, efficiency of public debt becomes a more relevant issue if the rate of interest is increasing.

Let us now turn to the case with a fixed interest rate and a budget deficit ratio target \( \bar{\mu} \). As in the case of a variable interest rate the tax rate is adjusted accordingly:

\[
\tau^H = 1 - \frac{\bar{\mu}}{s^g_{\text{pr}}} \left( 1 + \frac{\tau_B b^\mu}{\bar{\mu}} \right)^{-1}
\]

(230).

In this case the growth rate of public debt falls with the debt ratio:

\[
\hat{B}^\mu = \frac{\mu}{B^\mu}
\]

(231),
as does the primary-deficit-to-public-debt ratio:
As differential equation for the debt ratio along the SSL we can derive:
\[
\dot{b}_{\text{SSL}} = \alpha \mu - \alpha (\delta + n)b_{\text{SSL}}
\]  
(233)

which is identical to eq. (151), i.e. the differential equation for the debt ratio along the SSL in the case of an endogenous rate of interest on government bonds. Hence, other than in case of a fixed tax rate, with a budget deficit ratio target and correspondingly adjusted tax rate it does not matter for (relative) stability of the steady state whether the rate of interest is variable or not, as the changes in the tax rate just compensate this impact on the budget deficit, and hence on the steady state debt ratio:

\[
\frac{\mu}{\delta + n}; \frac{\partial \tilde{B}^*}{\partial \delta} < 0; \frac{\partial \tilde{B}^*}{\partial \delta} = 0
\]

whose size depends on the growth rate, but not on the interest rate. The sign of the steady state primary deficit ratio depends on whether the interest rate exceeds the growth rate or not:

\[
\tilde{d}^* = \tilde{\pi} \left(1 - \frac{T_B}{\delta + n}\right); \frac{\partial \tilde{d}^*}{\partial \delta} > 0; \frac{\partial \tilde{d}^*}{\partial \delta} < 0
\]

According to eqs (57) and (234) the steady state debt ratio \(\tilde{B}^*\) is efficient if the following inequality holds:

\[
T_B \leq \frac{1 - \alpha - \beta}{\mu} \frac{\delta + n}{\delta + n} \]  
(236)

From eq. (236) we can derive a maximum budget deficit ratio target that is not to be exceeded for public deficits to be efficient:

\[
\frac{\mu_{\text{Max}}}{\mu_{\text{Max}}} = \frac{1 - \alpha - \beta}{T_B} \left(\delta + n\right) \]  
(237)

According to eq. (237) \(\mu_{\text{Max}}^{\text{eff}}\) increases with the steady state growth rate of national income and falls with the rate of interest. For the steady state debt ratio \(\tilde{B}^*\) to be both long term sustainable and efficient, the following inequalities must hold:

\[
0 < \delta + n < T_B \leq \frac{1 - \alpha - \beta}{\mu} \frac{\delta + n}{\delta + n} \]  
(238)

This requires the following condition to be satisfied:

\[
\mu < 1 - \alpha - \beta \]  
(239)

As in the case of an endogenous interest rate the ratio of public consumption to national income is constant on and off the SSL, as it is again determined by eq. (174). Hence, in order to fulfill the liquidity constraint, as in the case of a variable interest rate, the deficit ratio target has to satisfy:

\[
\tilde{\mu} = \tilde{\mu}_{\text{Max}}^{\text{eff}} = \frac{s_g}{1 - (1 - \lambda) s_g \tilde{\mu}_{\text{Max}}^{\text{eff}}}; \frac{\tilde{\mu}_{\text{Max}}^{\text{eff}}}{\lambda g} > 0; \frac{\partial \tilde{d}^*}{\partial \delta} < 0
\]

(240)
With a budget deficit ratio target trace, determinant, and discriminant of the Jacobian matrix of the corresponding system of differential equations for the private and public capital-to-labor ratios are identical to the ones in the cases of an endogenous rate of interest on government bonds (see eqs (141) - (143)).

(a) Budget deficit ratio target \((\bar{\eta} = 0.01)\)

(b) Primary surplus ratio target \((\pi = -0.01)\)

Figure 28: Budget deficit ratio target and primary surplus ratio target compared: steady state values of the shadow rate of return on public debt, of the debt ratio, of the budget deficit ratio, of the government’s interest payments ratio, of the primary deficit ratio, and of the difference between public debt growth and interest rate depending on the steady state growth rate

\[
\begin{align*}
\delta & = 0.02, \sigma_{\pi} = 0.1, \gamma = 1, \alpha = 0.7, \beta = 0.2, \lambda = 1, T_B = 0.02
\end{align*}
\]

Hence, the steady state remains a stable node, as long as the steady state growth rate of national income is positive, independently of whether the solvency constraint is satisfied or not. As the steady state growth rate of national income and the rate of interest are beyond government’s control, in case that \(\delta + n > T_B > 0\), the government’s solvency constraint can only be satisfied in case of a budget deficit ratio target if it is set to or below zero, i.e. if a debt brake is introduced or a budget surplus targeted. But this would imply a tax rate of unity or larger, as long as the households’ propensity to invest is positive.

In Figure 31 (a) the light gray area shows the combinations of the interest rate and the budget deficit ratio target for which both the liquidity and the solvency constraint as well as the efficiency condition are fulfilled.

If, instead, the primary deficit ratio \(\pi\) is targeted by varying the tax rate accordingly we get:

\[
\begin{align*}
\pi & = \frac{s^{\pi}_{pr} - \pi - (1 - s^{\pi}_{pr}) T_B b^{\pi}}{s^{\pi}_{pr} \left(1 + T_B b^{\pi}\right)}; \quad \frac{\partial \pi}{\partial (-\pi)} > 0, \quad \frac{\partial \pi}{\partial T_B} < 0, \quad \frac{\partial \pi}{\partial b^{\pi}} < 0
\end{align*}
\]

Contrary to the other cases, the growth rate of public debt rises with the debt ratio if a primary surplus \((\pi < 0)\) is targeted:

\[
\dot{B}^{\pi} = \frac{\pi}{b^{\pi}} + r_B
\]

as does the primary-deficit-to-public–debt ratio:

\[
\frac{d^{\pi}}{db} = \frac{\pi}{b^{\pi}}
\]

The growth rate of public debt is positive for

\[
B^{\pi} > -\frac{\pi}{r_B}
\]
As differential equation for the debt ratio along the SSL we can derive:

\[ \dot{b}\pi^{SSL} = \alpha \pi - \alpha (\delta + n - \tau_B) b\pi^{SSL} \] (245)\(^{51}\)

with the following steady state value for the debt ratio:

\[ \ddot{b}\pi^{*} = \frac{-\pi}{\tau_B - (\delta + n)} ; \quad \frac{\partial \ddot{b}\pi^{*}}{\partial \pi^*} > 0, \quad \frac{\partial \ddot{b}\pi^{*}}{\partial \tau_B} < 0, \quad \frac{\partial \ddot{b}\pi^{*}}{\partial (\delta + n)} > 0 \quad \forall \pi < 0 < \delta + n < \tau_B \] (246)

which is positive and unstable if \( \pi < 0 < \delta + n < \tau_B \), i.e. if the solvency constraint is satisfied. Hence, with a primary surplus ratio target, whether the rate of interest is variable and tied to the rate of return on private capital via a constant spread factor or fixed, is decisive for stability. The steady state debt ratio increases with the primary surplus ratio target and the steady state growth rate, as long as \( 0 < \delta + n < \tau_B \). Thus, the signs of the impacts of increases of the primary surplus ratio target and the steady state growth rate on the steady state debt ratio are just opposite to those in the case with a primary surplus ratio target, yet endogenously determined rate of interest on government bonds (see eq. (207)), where the steady state is stable for \( 0 < \delta + n < \tau_B^* \). With a fixed rate of interest the steady state debt ratio is efficient according to eq. (57) if the following inequality holds:

\[ \tau_B \leq \frac{(1 - \alpha - \beta)(\delta + n)}{\pi + (1 - \alpha - \beta)} \quad \forall (1 - \alpha - \beta) > -\pi \] (247).

For \( 0 < \delta + n < \tau_B \) inequality (247) can only be satisfied if a surplus ratio is targeted:

\[ \pi < 0 \] (248).

Furthermore, for the government to be solvent \( (0 < \delta + n < \tau_B) \) in a steady state with a positive debt ratio the primary surplus has to be smaller than the government’s interest payments, as we get from eq. (246) and inequality (248):

\[ 0 < -\pi < \tau_B \ddot{b}\pi^{*} = \tau_B \frac{-\pi}{\tau_B - (\delta + n)} \] (249),

which is in line with inequality (100).

According to eqs (57) and (246), we see that the steady state shadow rate of return on public debt falls with the growth rate. Hence, there is a maximum growth rate beyond which the efficiency condition is no longer satisfied:

\[ (\delta + n)^{\pi_{Max}} = \tau_B \frac{\pi + 1 - \alpha - \beta}{1 - \alpha - \beta} \] (250).

A comparison of Figure 25 (a) and Figure 28 (a) and (b) shows the similarity of the cases with a fixed tax rate and a budget deficit ratio target on the one hand and the stark differences with the case of a primary surplus ratio target on the other. In this last case, as the steady state growth rate approaches the predetermined rate of interest from below, the steady state values of the debt ratio, the budget deficit ratio, and the government’s interest payments ratio explode, an obvious sign of public debt’s unsustainability. Qualitatively, a primary surplus ratio target with a fixed rate of interest is the special case considered by Gärtner and de Grauwe. The only (quantitative) difference is that, along the SSL, the right hand side is multiplied by labor’s elasticity of production.

\(^{51}\) The corresponding elasticities are:\( e_{Y_B}^{SSL,*} = \frac{1 - \alpha}{\delta + n} (\delta + n - \tau_B) \) and \( e_{\tau_B}^{SSL,*} = 0 \).
In case of a primary surplus target the dynamics crucially depend on whether the rate of interest on government is larger or smaller than the steady state growth rate: (i) For \( r_B < \delta + n \) the steady state is a stable node. Private and public capital as well as public debt are negative in the steady state. This would require a tax rate exceeding 100 %, as long as the households are willing to invest a share of their disposable income in government bonds, obviously a highly unrealistic case.

(ii) For \( r_B > \delta + n \) the steady state is a saddle point with a primary surplus ratio target. The determinant of the Jacobian matrix of the corresponding system of differential equations for the private and public capital-to-labor ratios is negative, the discriminant positive, and the trace positive, zero or negative. The SSL is the unstable saddle path. This instability can also be seen from the differential equation for the debt ratio along the SSL (see eq. (245)). For \( r_B > \delta + n \) the slope of the phase curve is positive. Accordingly, fixing the rate of interest at a level higher than the steady state growth rate makes the steady state unstable. If the initial debt ratio exceeds the steady state value the debt ratio grows forever, approaching the growth rate \( \alpha (r_B - \delta - n) \). According to eqs (147) and (245) we get

\[
\hat{\gamma}_{SSL} = \alpha (\delta + n) + (1 - \alpha) \left( r_B + \frac{\pi}{d\pi_{SSL}} \right)
\]  

(251).

For \( \pi < 0 \) and an ever growing debt ratio along the SSL, the growth of national income approaches the weighted average of its steady state value \( (\delta + n) \) and the rate of interest from below, whereas the growth rate of public debt approaches the rate of interest on government bonds, again from below. 52

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\[ \text{Figure 29: Primary surplus ratio target: debt ratio, change of debt ratio, growth rate of national income, difference between growth rates of public debt and national income, and growth rate of growth rate of public debt depending on the growth rate of public debt along the SSL} \]

\[ (s_{pr} = 0.02; s_{pr} = 0.1; \gamma = 0.7; \alpha = 0.2; \lambda = 0.1; \delta = 0.02; n = 0.0) \]

A comparison of Figure 13 (b)53 and Figure 29 (a) shows the stability of the steady state in the case of a variable rate of interest and the instability of the steady state if the exogenous rate of interest exceeds the steady state growth rate of national income, as the slope of the curve showing the growth rate of the growth rate of public debt as a function of the growth rate of public debt is negative in the former steady state and positive in the latter. In the latter case, once the growth rate of public debt exceeds its steady state value it approaches the rate of interest, and the debt ratio tends to infinity.

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52 Spahn 2016 (p. 127) considers the special case of a primary surplus being equal to zero. Hence, the steady state debt ratio vanishes, too. Its stability depends on the rate of interest being smaller or larger than the steady state growth rate.

53 Figure 13 (b) is also relevant for a budget deficit ratio target and a primary deficit ratio target with a variable rate of interest, as the steady state loci do not differ.
Figure 30: Exogenous rate of interest on government bonds: phase diagrams and stream plots

\( T_B = 0.035; \delta_B = 0.02; \alpha = 0.1; \gamma = 1; \alpha = 0.7; \beta = 0.2; n = 0; \delta = 0.02; \lambda = 1 \)
Figure 29 (b) shows the case where the exogenous rate of interest is smaller than the steady state growth rate of national income. Once the growth rate of public debt is smaller than the rate of interest it tends to zero.

If the debt ratio tends to infinity as in Figure 29 (a), the present value of public debt does not approach zero, but a positive constant value (see Figure 30 (h)), as according to eq. (110) we get:

\[
\lim_{t \to \infty} \frac{PVB(t)}{\pi^{\text{SSL}}} = \lim_{t \to \infty} \frac{\pi}{B^{\text{SSL}(t)}} = 0
\] (252).

Once a shock occurs that makes the debt ratio larger than its steady state level, public debt is no longer long term sustainable despite a primary surplus, due to the instability of the steady state, hence due to its medium term unsustainability. This unsustainability does not only show up for very large debt ratios, but at the latest when the liquidity constraint can no longer be fulfilled, as public consumption would become negative when the government’s investment and interest expenditures exceed its revenue from taxes and budget deficit (see the crossing of the black phase curve and the orange \( \frac{C_g}{Y} \pi = 0 \) — locus in Figure 30 (b) as well as Figure 34).

For the ratio of public consumption to national income we can derive in general:

\[
\left( \frac{C_g}{Y} \right)^\pi = 1 - \frac{1 - s^g_{pr}(1 - \lambda)}{s^g_{pr}} (\pi + \tau B \pi^{\text{SSL}}); \quad \frac{\partial (\frac{C_g}{Y})^\pi}{\partial (\pi)} > 0; \quad \frac{\partial (\frac{C_g}{Y})^\pi}{\partial B^{\pi}} < 0; \quad \frac{\partial (\frac{C_g}{Y})^\pi}{\partial B^{\pi}} < 0 \forall \tau B > 0 (253),
\]

and for the steady state in particular by taking into account eq. (246):

\[
\left( \frac{C_g}{Y} \right)^{\pi^*} = 1 + \pi \frac{1 - s^g_{pr}(1 - \lambda)}{\tau B - (\delta + n)}; \quad \frac{\partial (\frac{C_g}{Y})^{\pi^*}}{\partial (\pi)} > 0; \quad \frac{\partial (\frac{C_g}{Y})^{\pi^*}}{\partial B^{\pi^*}} > 0 \forall \pi < 0 < \delta + n < \tau B (254).
\]

From eq. (253) we see that the ratio of public consumption to national income (i) increases with an increase of the primary surplus ratio target and a decrease of the interest rate for a given debt ratio, and (ii), for a given primary surplus ratio target and a given interest rate, decreases with the debt ratio and finally vanishes for:

\[
\bar{b}_{\text{Max}}^{\pi} \left[ \left( \frac{C_g}{Y} \right)^{\pi^*} \right] = 0 \quad \Rightarrow \quad 1 + \frac{s^g_{pr}(1 - \lambda)}{1 - s^g_{pr}(1 - \lambda)}; \quad \frac{\partial B_{\text{Max}}^{\pi^*}}{\partial (\pi)} = 1 < 0 \forall \tau B > 0 (255),
\]

the maximum debt ratio at which the liquidity constraint is satisfied. It is the larger the larger the primary surplus ratio target. Hence, for a given debt ratio exceeding \( s^g_{pr} / (1 - s^g_{pr}(1 - \lambda)) \tau B \) a sufficiently high primary surplus ratio target is necessary in order to satisfy the government’s liquidity constraint.

From eq. (254) we can compute the maximum primary surplus ratio target for which the liquidity constraint is just satisfied in the steady state:

\[
(-\pi) \left[ \left( \frac{C_g}{Y} \right)^{\pi^*} \right] = 0 = (-\pi)^{\text{Max}}_{\text{eq}} \left( \frac{s^g_{pr}(\tau B - \delta - n)}{(\delta + n)(1 - s^g_{pr}(1 - \lambda))} > 0 \forall 0 < \delta + n < \tau B (256),
\]
which is the larger, the larger the rate of interest. As the steady state debt ratio increases with the primary surplus ratio target there is a maximum threshold of the latter that is not to be exceeded in order to satisfy the liquidity constraint in the steady state (see the blue solid line in Figure 31 (b)).

In case of an ever increasing debt ratio public debt becomes inefficient, once the debt ratio exceeds

$$\pi_{\text{eff}} = \frac{1 - \alpha - \beta}{\tau_B}$$  \hspace{1cm} (257),

$$1 - \alpha - \beta$$

$$(1 - \alpha - \beta)(\tau_B - \delta - n) > 0 \ \forall \ \delta + n < \tau_B$$  \hspace{1cm} (258),

$$-\pi + \frac{s^{g}_{pr}}{1 - s^{g}_{pr}(1 - \lambda)} = 1 - \alpha - \beta$$  \hspace{1cm} (259),
For a vanishing primary surplus ratio target \((\pi = 0)\), which implies \(\tau_B = \delta + n\), this condition is satisfied if the Golden Rule of Public Debt Accumulation (GRA) is followed as we can derive (see for details Englmann 2016)

\[
\left( \frac{s_{pr}^g}{s_{pr}} \right)^{GRA} = 1 - \alpha - \beta
\]

(260),

\[
\lambda^{GRA} = 1
\]

(261).

The light gray area in Figure 31 (b) shows the set of combinations of the exogenously given interest rate and the primary surplus ratio target that imply a steady state debt ratio that satisfies the liquidity and the solvency constraint as well as the efficiency condition, but not the stability condition. If we compare Figure 31 (b) with Figure 20 (b) we notice that in Figure 31 (b) we have maximum thresholds of the primary surplus ratio target, whereas we have minimum thresholds in Figure 20 (b). This is due to the fact that the steady state public consumption ratio falls with the primary surplus ratio target in case of a fixed interest rate (see eq. (254) and bb). In case of a variable rate of interest an increase in the primary surplus ratio target leads to an increase in the steady state tax rate, whereas the government’s steady state interest payments ratio is unchanged, thus widening the fiscal space for financing public consumption. With a fixed interest rate an increase in the primary surplus ratio target leads to a decrease in the steady state tax rate (see eq. (262))

\[
\frac{s_{pr}^g - \pi - (1 - s_{pr}^g)}{\tau_B - (\delta + n)} \leq \frac{1}{\delta - \pi} \frac{(\delta + n)}{s_{pr}^g \left[ (1 - \pi) \tau_B - (\delta + n) \right]^2} < 0 \quad 0 < \delta + n < \tau_B
\]

(262),

whereas the government’s steady state interest payments ratio increases due to a rise in the steady state debt ratio, thus reducing the fiscal space for financing public consumption (see eq. (263))

\[
\frac{\tau_B B^{\pi^*}}{\tau_B - (\delta + n)} \leq \frac{(\delta + n)}{\tau_B - (\delta + n)} > 1
\]

(263).

The government’s liquidity constraint is satisfied on and below the \(b_{\text{Max}}^{\pi} \left[ (C_g/Y)^\pi = 0 \right] \) plain in Figure 32.

Figure 32: Primary surplus ratio target and fixed rate of interest: Maximum debt ratio satisfying the government’s liquidity constraint, steady state and ‘actual’ debt ratio

\[ \left( s_{pr}^g = 0.03; s_{pr}^f = 0.1; \gamma = 1; \alpha = 0.7; \beta = 0.2; \lambda = 1; \delta = 0.02; n = 0.0; \epsilon = 0.9 \right) \text{ Exotaupirbex: x:2.4,y:-0.5,z:0.7} \]

In Figure 32, by assumption, the ‘actual’ debt ratio is 100%. If we additionally assume that the rate of interest is four percent and hence the steady state is unstable we can distinguish three ranges of the
primary surplus ratio target: (i) for sufficiently low values the actual debt ratio is so high that it exceeds both the maximum debt ratio satisfying the government’s liquidity constraint and the steady state debt ratio. Hence, the government’s liquidity constraint is not fulfilled and the debt ratio tends to grow without limit. (ii) For medium values of the primary surplus ratio target the actual debt ratio is in a range such that it is lower than the maximum debt ratio satisfying the government’s liquidity constraint, yet higher than the steady state debt ratio. Hence, the government’s liquidity constraint is initially fulfilled, but the debt ratio still tends to grow without limit until the government’s liquidity constraint is violated again. (iii) For high values of the primary surplus ratio target the actual debt ratio is in a range such that it is lower than both the maximum debt ratio satisfying the government’s liquidity constraint and the steady state debt ratio. Hence, the government’s liquidity constraint is fulfilled and the debt ratio tends to fall.

If we apply this reasoning to the case of Greece we see that in case of a given interest rate and a given debt ratio the so-called Troika, consisting of the European Commission, European Central Bank and The International Monetary Fund, has to request a sufficiently large primary surplus ratio target and hence according to eq. (241) a sufficiently high tax rate if it wants to reduce the Greek public debt ratio. Yet, Figure 32 (b) also shows that a debt relief and/or interest rate relief lower the required primary surplus ratio target and hence the required tax rate.

From Table 1, Table 2 and Table 4 we see that only in case of a fixed interest rate and a primary surplus ratio target there exists no parameter combination for which both the stability conditions and the solvency constraint are satisfied in the steady state. In all the other cases the stability conditions and the solvency constraint can be fulfilled simultaneously. But even if the stability conditions and the solvency constraint cannot be satisfied simultaneously in case of a fixed interest rate and a primary surplus ratio target in the steady state this does not mean that there exists no range of the public debt ratio for which both the liquidity constraint and the solvency constraint are fulfilled - the latter in the sense that the rate
of change of public debt is smaller than the rate of interest – and that the debt ratio declines. This is the case for

$$\bar{b}^\pi = -\frac{\pi}{r_B - (\delta + n)} > \bar{b}_\text{Max} \left[ \frac{C_g}{Y} \right] = 0 = \frac{1}{r_B} \left( -\frac{\pi + s_{pr}}{1 - s_{pr}(1 - \lambda)} \right) > \bar{b}^\pi > -\frac{\pi}{r_B}$$  \hspace{1cm} (264),

i.e. for a debt ratio smaller than the steady state value yet larger than the debt ratio for which public debt ceases to increase which would imply a tax rate of unity. But, once the actual debt ratio exceeds the steady state debt ratio, reaching this range requires either a tax increase in order to realize a higher primary surplus ratio target or the cooperation of the creditors, in the form of a debt relief or an interest rate relief. Once this range is reached, the debt ratio declines at an increasing rate according to eq. (245) and would become smaller than the range’s lower threshold according to eq. (264) implying a tax rate exceeding unity. Hence, a policy targeting a fixed primary surplus ratio would become unsustainable for an ever decreasing debt ratio, too.

If we compare the slopes of the phase curves for the debt ratio we get:

$$\frac{\dot{b}^\pi}{\dot{b}_\text{SSL}} = -\alpha(\delta + n) < \frac{\dot{b}^\pi}{\dot{b}_\text{SSL}} = -\alpha(\delta + n - s_{pr}(1 - \tau)r_B) < \frac{\dot{b}^\pi}{\dot{b}_\text{SSL}} = -\alpha(\delta + n - r_B)$$  \hspace{1cm} (265).

Accordingly, with respect to stability and hence long term sustainability, it is advisable to target a (sufficiently low) budget deficit ratio, not the corresponding primary surplus ratio.

The phase curves of the debt ratio along the (identical) SSL for the three cases are shown in Figure 30 (a). We see that, for a sufficiently small propensity to invest in government bonds, they are almost identical for the case of a fixed tax rate and the corresponding budget deficit ratio target. Hence, in Figure 30 (c) – (h) only the cases of a budget deficit target and of a primary surplus target are shown.

![Figure 30](image)

**Figure 30:** Budget deficit ratio target and primary surplus ratio target: impact of a change in the rate of interest on the growth rate of public debt, the phase curve of the debt ratio, and the primary-deficit-to-public–debt ratio

$$s_{pr} = 0.02; s_{pr} = 0.1; \gamma = 1; \alpha = 0.7; \beta = 0.2; \pi = 0; \delta = 0.02; \lambda = 1$$

Figure 33 shows the impact of a change in the rate of interest on the growth rate of public debt, the phase curve of the debt ratio, and the primary-deficit-to-public–debt ratio both for a budget deficit ratio target and a primary surplus ratio target. The differences between the two cases are significant: With a budget deficit ratio target only the primary-deficit-to-public–debt ratio is affected, whereas with a primary surplus ratio target only the primary-deficit-to-public–debt ratio is not affected. With a budget deficit ratio target an increase in the rate of interest leaves the dynamics of public debt unchanged, yet has a positive impact on the primary surplus. With a primary surplus ratio target, an increase in the rate of interest
increases the growth of public debt by the same amount. As the steady state debt ratio declines, the actual debt ratio may very well be larger than the new steady state level, which means that the debt ratio grows forever and the growth rate of public debt approaches the higher rate of interest which leads to a violation of the solvency constraint despite the primary surplus ratio target. With a budget deficit ratio target, the rate of interest should exceed the growth rate of national income in any case in order to make public debt long term sustainable. In case of a primary surplus ratio target the answer is not so clear-cut, as, with an exogenous interest rate, public debt is not sustainable in any case. And a decrease of the interest rate would at least lower the growth rate of public debt.

If we compare the three differential equations for the debt ratio we see that the rate of interest has no impact in the case of a given budget deficit target:

$$\frac{\partial b_\mu^{SSL}}{\partial \tau_B} = 0$$

(266).

It has a limited impact in the case of a given tax rate

$$\frac{\partial b^{SSL}}{\partial \tau_B} = \alpha s^{SSL}_P (1 - \tau) b^{SSL} > 0 \forall b^{SSL} > 0$$

(267)

and a major impact in the case of a given primary deficit target

$$\frac{\partial b^{\pi^{SSL}}}{\partial \tau_B} = \alpha \bar{b}^{\pi^{SSL}} > 0 \forall \bar{b}^{\pi^{SSL}} > 0$$

(268)

Thus, we may conclude that, in terms of robustness with respect to interest rate shocks, too, it is prudent to fix the tax rate or target the budget deficit ratio, but not the primary surplus ratio.

In case of the Greek sovereign debt crisis it has been intensely discussed whether a debt relief is necessary in order to make the Greek sovereign debt sustainable or whether a sufficiently low rate of interest in combination with structural reforms suffices. Successful structural reforms increase the rate of technical progress $\delta$. As surplus ratio targets were agreed upon in the Second Economic Adjustment Programme for Greece in 2012 and in the Memorandum of Understanding between the European Commission Acting on Behalf of the European Stability Mechanism and the Hellenic Republic and the Bank of Greece on 19 August 2015 we will discuss this question for a surplus ratio target with a rate of interest exceeding the steady state growth rate of national income first.

In case public debt is unsustainable if the actual debt is larger than the steady state debt ratio, as this leads to an ever increasing debt ratio. The public debt ratio does not increase anymore without limit, once the actual debt ratio $b_0$ is smaller than or equal to its steady state value $\bar{b}^{\pi}$. This can be achieved by reducing the actual debt ratio by means of a sufficiently large debt relief leading to a sufficiently large reduction of the debt ratio (db):

$$db \geq db^{Min} = b_0 - \bar{b}^{\pi}$$

(269).

Another possibility is to increase the steady state debt ratio beyond its actual value. For $\tau_B > \delta + n$ this can be achieved by three measures according to eq. (246): (i) an increase in the target surplus ratio, (ii) an increase in the rate of technical progress and (iii) a reduction of the rate of interest on government bonds. In case (ii) and (iii) the changes have to be sufficiently small such that $\tau_B > \delta + n$ remains valid.

In case of a reduction of the rate of interest on government bonds we get for the induced change of the steady state debt ratio:

$$d\bar{b}^{\pi} = \frac{\bar{b}^{\pi}}{\delta + n - \tau_B} d\tau_B \geq b_0 - \bar{b}^{\pi} \forall 0 > d\tau_B > \delta + n - \tau_B^0$$

(270).
and hence we can determine the minimum change of the rate of interest on government bonds $d r_B^{\text{Min}}$ that just suffices to increase the steady state debt ratio to the initial level of the debt ratio $b_0$:

$$-d r_B^{\text{Min}} = \frac{b_0 - b_0^n}{b_0^n} (r_B - \delta - n) \quad \forall \ 0 > d r_B^{\text{Min}} > \delta + n - r_B$$

(271).

Finally, from eqs (269) and (271) follows:

$$- \frac{d r_B^{\text{Min}}}{r_B} = db^{\text{Min}}$$

(272).

According to eq. (272) the present value of the minimum interest payment reduction just equals the minimum debt relief. Hence, the two are economically equivalent.

A debt relief implies a movement both along the unchanged phase curve of the debt ratio and along the unchanged tax curve (see eq. (241), whereas an interest rate relief leads to a shift of both the phase curve and the tax curve. Figure 34 shows the case of a debt relief reducing the debt ratio to its steady state level and of the economically equivalent interest rate relief. Hence, with a primary surplus ratio target both a debt relief and an interest rate relief have to be accompanied by a tax increase (see Figure 34) which is larger in the case of a debt relief.

![Figure 34: Primary surplus ratio target: minimum debt relief and equivalent minimum interest rate relief](image)

According to eq. (272) the present value of the minimum interest payment reduction just equals the minimum debt relief. Hence, the two are economically equivalent.

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Figure 34: Primary surplus ratio target: minimum debt relief and equivalent minimum interest rate relief

Obviously, having shown the economic equivalence of a debt relief and an adequately sized interest rate relief does not help to decide among the two alternatives from an economic point of view. Instead, it suggests that a combination of the two might be advisable if the preferences of the government and its creditors diverge, e.g. for political reasons. Hereby it has to be emphasized that the more successful the structural reforms are and hence the higher the rate of technical progress, the less room exists to reduce the rate of interest, if the solvency constraint is to be satisfied. This is an argument in favor of a debt relief. An argument against a debt relief is the problem of moral hazard that may be created if a debt relief is granted, especially in a country like Greece, whose recent history is a history of generous debt reliefs, too (Kalyvas 2015). Such a moral hazard problem does not exist, at least not to the same extent, in case of an interest rate relief, which can easily be reversed.

In case of a budget deficit ratio target the steady state is stable, independently of the rate of interest on government bonds exceeding the steady state growth rate or not. Hence, in any case public debt is...
medium term sustainable. Its long term sustainability depends on the rate of interest on government bonds being larger than the steady state growth rate. If this condition is fulfilled a debt relief is not necessary, an interest rate relief would even be harmful if it led to the violation of the solvency constraint.

This raises the question what the impact of ultra-low or negative interest rates on government bonds is (see Hannoun 2015) in the presented model. Let us start with the case of a zero rate of interest. Here we have:

\[
s_{pr}^0 (1 - \tau) = \frac{\dot{B}}{Y} = \mu = \delta = \pi > 0 \quad \land \quad s_{pr}^0 (1 - \tau) > 0 \quad (273).
\]

Budget deficit and primary deficit coincide, and hence the cases of a budget deficit ratio target and a primary deficit ratio target, too. As long as the households’ propensity to invest in government bonds is positive, the tax rate smaller than unity, and \( \delta + n > 0 \) the steady state debt ratio is positive and stable.

But the government’s solvency constraint is violated, as the present value of government debt tends to infinity because of

\[
\overline{PVB} = \dot{\delta} = \delta + n > 0 \quad (274).
\]

This scenario is shown in Figure 35 (a) and (b). With a zero rate of interest the government’s solvency constraint can only be fulfilled if both the primary deficit and the budget deficit ratio are negative or zero. But the former would lead to an ever growing present value of the government’s claims on private households (see Figure 35 (c) and (d)), a situation that is not sustainable from the point of view of pri-
vate households. Hence, with a positive growth rate and a zero rate of interest on government bonds, sustainability requires a zero budget deficit ratio target, hence a debt brake.

Hence, with a zero rate of interest only a zero propensity to invest in government bonds \( s_{pr}^G (1 - \tau) \), a zero budget deficit ratio, a zero primary deficit ratio, and a zero government debt in the long run are sustainable whereas (positive) public debt is either medium term or long term unsustainable.

In a shrinking economy \( \delta + n < 0 \) the government’s solvency constraint would be satisfied with a zero rate of interest on government bonds. But the equilibrium would be unstable and the propensity to invest in government bonds after tax \( s_{pr}^G (1 - \tau) \) would have to be negative. This shows that the model presented cannot adequately deal with a shrinking economy and with a decreasing public debt in absolute terms, but only with a growing economy and a decreasing debt-to-GDP ratio, hence a shrinking public debt in relative terms. Nonetheless, Figure 36 shows a scenario with a negative steady state growth rate, a larger, yet negative rate of interest and a primary surplus ratio target. Public debt as well as its present value approach zero in the unstable steady state, and at a still faster rate if the debt ratio’s initial value is smaller than its steady state level. If the debt ratio’s initial value is larger than its steady state level, the rate of change of public debt approaches the rate of interest, and hence the present value of public debt a constant positive level.

At first sight, with a negative rate of interest on government bonds a primary surplus does not seem to be necessary to pay back government debt, as the government’s interest revenue can be used to this end (see Figure 36 (d)). But for the solvency constraint to be satisfied the growth rate of public debt has
to be smaller than the interest rate, as condition (111) has to be fulfilled with a negative rate of interest, too. Hence if \( \lim_{t \to \infty} b(t) \geq 0 \), \( \lim_{t \to \infty} d(t) \) has to be negative, implying a primary budget surplus.

7. Concluding Remarks

The question that motivated the presented research was whether in general public debt can be sustainable in the sense that the equilibrium debt ratio can be stable and, simultaneously, the government’s solvency constraint can be satisfied for a set of parameter constellations. To this end a simple neoclassical growth model of the Solow type (Solow 1956) was developed that includes government debt. With the help of this model it was shown that public debt can be sustainable for a set of parameter constellations. This set of parameter constellations can be further divided into two subsets: for one the Maastricht criterion for the public debt ratio is fulfilled, for the other not. Thus, the Maastricht criteria for the public deficit and the public debt ratio cannot be justified by theoretical considerations of public debt sustainability. What can be justified instead as a criterion of public debt sustainability is a primary surplus. But it was shown that targeting a fixed primary surplus-to-national income ratio may slow down adjustment to the steady state, compared to a fixed budget deficit-to-national income ratio. Hence, the policy conclusion is to target a budget deficit ratio that is low enough in order to imply a primary surplus in the steady state. This holds especially if the rate of interest is fixed exogenously and larger than the growth rate. In this case, targeting a primary surplus ratio destabilizes the steady state and makes public debt unsustainable.

Finally one might wonder whether it is really adequate in the context of the models presented above to consider the government only to be solvent if the present value of its debt tends to zero in the long run. This requires that government can repay all its current debt through future primary surpluses in case that it cannot issue any new debt in the future. But, if all economic agents have rational expectations that, forever, private households are willing to invest a constant fraction of their disposable income in newly issued government bonds and hence to hold the same constant fraction of their wealth in government bonds it might be enough for the government to be considered solvent if the debt-to-GDP ratio tends to a constant positive value. Then medium and long term sustainability of public debt would coincide.

A shortcoming of the analysis is that the propensities to save and the safety discount factor are treated as exogenous parameters. Given decreasing marginal returns to capital and a positive sum of the rate of technical progress and the natural rate of labor supply, these two assumptions lead to the global stability of the steady state. Preliminary results of current research show that the endogenization of the propensities to save and the safety discount factor may destabilize the steady state. More general restrictions of the analysis are that we do not deal with a monetary and open economy including a financial sector that creates money through credit creation. The investigation of whether the introduction of a money creating financial sector changes the results presented above is left for future research.

REFERENCES


