Can Public Debt Be Sustainable?  
- A Contribution to the Theory of the Sustainability of Public Debt

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Abstract

The sustainability of government’s financial situation has two aspects: solvency and liquidity. The government is solvent if it satisfies its intertemporal budget constraint. In the steady state of a growing economy, the government is solvent if the public-debt-to-GDP ratio is constant and the real interest rate on government debt exceeds the growth rate of real GDP. The government is liquid if its instantaneous budget constraint is satisfied. According to Gärtner 2009, for a constant primary-deficit-to-GDP ratio, this steady state debt-to-GDP ratio is unstable if the real interest rate on government debt exceeds the growth rate of real GDP. Hence, there seems to be a tension between sustainability concerning government’s solvency and sustainability concerning government’s liquidity if the latter is defined as stability of the steady state debt-to-GDP ratio. This leads to the main research question of this paper: Is it true that in a growing economy a constant debt-to-GDP ratio can only be stable if the government’s solvency constraint is violated? Or put differently: Is it true that public debt cannot be sustainable? Obviously, one counterexample is enough to show that a proposition does not hold in general. Such a counterexample based on the Solow growth model is presented in the paper. Hence, in general, sustainability of public debt is possible in the sense, that both the solvency constraint and the liquidity constraint with a stable steady state debt-to-GDP ratio are satisfied for a set of parameter constellations.

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1. Introduction

Public debt and its sustainability have become an acute problem since the beginning of the Great Recession in various countries, especially in the Euro Area. As, among others, Kindleberger and Aliber 2011 or Reinhart and Rogoff 2009 have pointed out, this is by no means a new phenomenon in economic history. Yet, there seem to exist different views on what are the conditions for the sustainability of public debt or more specifically the public debt-to-GDP ratio.

The most fundamental concept of the sustainability of public debt seems to be that the government remains solvent, i.e. that it satisfies its intertemporal budget constraint, named in the following: solvency constraint. As follows e.g. from Romer 2012 p. 586f, this is the case if the real interest rate on real government debt is always positive and larger than the rate of growth of real government debt. In a growing economy, this means that the government’s solvency constraint is satisfied if the public-debt-to-GDP ratio is constant and the real interest rate on government debt exceeds the growth rate of real GDP. Thus, with respect to solvency, a condition for sustainable public debt is a sufficiently high real rate of interest compared to the growth rate of real public debt which, for a constant debt-to-GDP ratio, just equals the growth rate of real GDP.

Yet, during the Euro crisis high rates of interest on government bonds, especially high spreads relative to the German Bund were considered a sign of fiscal weakness, as governments of countries like Greece, Ireland, Italy, Portugal, and Spain had to offer high yields in order to stay liquid by selling their bonds in financial markets. A government is liquid if its instantaneous budget constraint\(^1\), named in the following liquidity constraint, is satisfied. In case of public deficits, satisfying the government’s liquidity constraint requires a sufficient demand for newly issued government bonds if monetization of public debt is excluded. The government’s instantaneous budget constraint translates into a differential equation for the debt-to-GDP ratio from which an equilibrium debt-to-GDP ratio can be computed (see e.g. Gärtner 2009 p. 394f and de Grauwe 2014 annex; in the following: G2). According to Gärtner 2009 p. 395f, for a constant primary-deficit-to-GDP ratio, this steady state debt-to-GDP ratio is unstable\(^2\) if the real interest rate on government debt exceeds the growth rate of real GDP. In this case a shock, that makes the actual debt-to-GDP ratio larger than its equilibrium value, leads to an ever increasing debt-to-GDP ratio and an ever increasing deficit-to-GDP ratio. In a closed economy, this increasing deficit-to-GDP ratio cannot be financed out of private agents’ savings if it approaches unity, and public debt becomes unsustainable, as the government becomes illiquid. Hence, fiscal sustainability concerning government’s liquidity in the sense of resilience to shocks requires the stability of the equilibrium debt-to-GDP ratio which in turn requires the growth rate of real GDP to exceed the real rate of interest on government debt for a given primary-deficit-to-GDP ratio.

Hence, there seems to be a tension between sustainability concerning government’s solvency and sustainability concerning government’s liquidity if the latter is defined as stability of the steady state debt-to-GDP ratio. This leads to the main research question of this paper: Is it true that in a growing economy a constant debt-to-GDP ratio can only be stable if the government’s solvency constraint is violated? An affirmative answer would imply that, by its very nature, public debt is unsustainable and, hence, has to be avoided. This would amount to a theoretical justification of a public debt brake.

Obviously, one counterexample is enough to show that a proposition does not hold in general. Such a counterexample is presented here. To this end the simple Solow growth model (Solow 1956) is generalized in order to include public capital and public debt as well as the government’s liquidity and solvency constraint. A growth model of the Solow type is chosen because its relevant steady state is asymptotically stable. Hence, the conditions under which the originally stable steady state becomes unstable can

\(^1\) In the following time is assumed to be continuous.

\(^2\) Now and in the following, by stable we mean asymptotically stable in the sense that after a shock a system returns to its equilibrium (see e.g. Gandolfo 2010 pp. 331ff). Contrary to e.g. European Central Bank 2011 p. 64, by stable we do not mean constant.
be studied. This research strategy obviously could not be followed if the starting point were a model whose steady state is either asymptotically unstable as a growth model of the Harrod-Domar type\(^3\) or a saddle point as an optimal growth model of the Ramsey type.

Whereas G2 treat the interest rate on public debt, the growth rate of GDP, and the primary-deficit-to-GDP ratio as exogenous variables, in the model presented below they will be endogenous. Here the stability of the (economically relevant) steady state and hence the equilibrium public debt ratio does not depend on whether the equilibrium growth rate of GDP exceeds the equilibrium rate of interest.

From an economic point of view the main difference between the model presented below and the model presented e.g. by G2 consists in the following: Whereas in G2’s model the debt dynamics are driven by the supply side (which might be called ‘Say’s Law of Public Finance’), namely the government’s need to finance a budget deficit, in the model presented below the debt dynamics are driven by the demand side, namely the demand of private households for government bonds, just as in the original Solow growth model where private capital accumulation is driven by the savings of private households and hence the demand of private households for stock or bonds issued by private firms in order to finance private investments in real capital.

In almost all countries in the world there has been public debt, at least during the last decades. In some countries sovereign debt crises were experienced, but luckily not in all. Hence in a “realistic” model for countries without sovereign debt crisis, that can describe and explain several stylized facts, there should exist steady states with a positive and stable debt-to-GDP ratio without violating the government’s solvency constraint. The latter requires the real rate of interest on public debt to exceed the growth rate of real GDP, which in turn requires a primary public budget surplus for a positive steady state debt-to-GDP ratio.

In the model presented below such steady states exist, as long as private households want to hold a part of their savings in government bonds as an asset with (under normal conditions) a positive nominal rate of return and comparatively low risk. As in the original Solow model money is not taken into account. Hence there cannot be any liquidity preference in the sense of a demand for money. But the demand for government bonds can be viewed as stemming from a sort of liquidity or safety preference which induces private households to accept a lower rate of return, i.e. a safety discount for investments in government bonds compared to investments in bonds or equities issued by private firms. How strong this safety preference can become one could observe during the Euro crisis, when the nominal yield on bonds issued by the Federal Government of Germany sometimes became negative.

Presenting a model with a positive steady state debt-to-GDP ratio, where the government is both solvent and liquid in the sense of resilience to shocks and, hence, where its public debt is sustainable, does not mean to deny the possibility of unsustainable public debt. The model presented can thus form the base for finding out deeper reasons for the occurrence of fiscal crises.\(^4\) In the model private households’ propensity to invest in government bonds is assumed to be an exogenous parameter just as the safety discount factor. These rather strong assumptions can be justified by three considerations: (i) they are in line with the original Solow growth model, where the saving rate is assumed to be exogenous; (ii) this is a paper on sovereign debt sustainability, not on sovereign debt crises including runs on government bonds; (iii) in another paper by the author both private households’ propensity to invest in government bonds and the safety discount factor will be endogenized.

The remainder of the paper is organized as follows: In section 2 the Solow growth model with public debt is presented. In section 3 we derive the system of differential equations, the steady state, and carry out various comparative dynamic analyses. In section 4 we will turn to the question whether it is really true that in a growing economy a constant debt-to-GDP ratio can only be stable and hence sustainable if the real rate of interest on government debt is lower than the growth rate of real GDP which in turn

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3 Domar 1944 is a seminal paper for the discussion of the sustainability of public debt in the context of economic growth.
4 First steps in this direction are undertaken in an accompanying paper by the author.
leads to a violation of the government’s solvency constraint. To answer this question we will first describe Romer’s approach (Romer 2012) and how it fits into the model presented here. Then we describe the approach by Gärtner 2009 and de Grauwe 2014 (G2), and finally we compare their approach with the one presented here.

In the model presented here the steady state and hence the steady state ratio of public debt to national income turn out to be stable under rather general conditions, namely, whenever the sum of the rate of technical progress and the rate of change of labor supply, i.e. the equilibrium growth rate of real GDP is positive and the partial production elasticity of labor in efficiency units lies within a range between zero and unity. Hence, stability does not depend on the sign of the difference between the growth rate and the rate of interest on government bonds. What really matters for the sustainability of public debt is not any more or less arbitrary level of the public-deficit-to-GDP ratio or of the public-debt-to-GDP ratio, but whether there is a primary surplus or deficit in the steady state. Section 5 concludes.


Following Aschauer 1989 we take account of public capital $K_g$ in the production function

$$Y^S = \gamma A^\alpha L^{1-\alpha} K^{1-\beta}$$

where $Y^S$ denotes aggregate supply of goods and services produced in the economy, $A$ technical efficiency, $L$ labor input, $K$ private capital. We assume a Cobb-Douglas production function with constant returns to scale in order to be able to carry out numerical analyses. The main results of the analysis do not depend on the specific form of the production function as long as constant returns are preserved and the production function has a representation of the form $Y^S = F(AL,K_{pr},K_g)$, i.e. technical progress can be represented as purely labor-augmenting (see Acemoglu 2009, pp. 58 ff.). For the sake of simplicity, the dependence of the variables on time is omitted in general. Total factor productivity grows at the rate of technical progress $\delta$, labor supply at the natural rate $n$. Labor supply is inelastic with respect to the real net wage. Capital and labor are fully employed:

$$A = A_0 e^{\delta t}; \quad \delta > 0$$

$$L = L_0 e^{nt}; \quad n > 0$$

Aggregate demand for goods and services consists of private ($C_{pr}$) and public consumption ($C_g$) as well as of net private ($I_{pr}$) and net public investment ($I_g$).

$$Y^D = C_{pr} + I_{pr} + C_g + I_g$$

As in the Solow model, the market for goods and services is in equilibrium:

$$Y^S = Y^D = Y$$

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5 This production function differs from the one e.g. in Barro 1990 and Minnea and Villieu 2012. These authors take ‘productive’ public expenditures into account, not public capital.
where \( Y \) denotes net domestic product which equals net national income, as we consider a closed economy.\(^6\) Private consumption depends on available income of households \((Y - T + r_B B)\)

\[ C_{pr} = c_{pr} (Y - T + r_B B) \]  

(6),

where \( c_{pr} \) denotes households' propensity to consume with respect to disposable income, \( T \) net transfer payments from households to government (taxes plus social contributions from private households minus government's transfer payments to private households), and \( r_B \) the rate of interest on government bonds \( B \). For households' propensity to save obviously holds:

\[ s_{pr} = 1 - c_{pr} \]  

(7),

and hence for private savings \( S_{pr} \)

\[ S_{pr} = s_{pr} (Y - T + r_B B) \]  

(8).

In the original Solow model neither financial markets nor a banking sector are explicitly modelled. It is simply assumed that the financial market for firms' (newly issued) bonds or equities is in equilibrium. Hence, (in the end) households' savings are just invested by private firms. Now we have two financial markets: one for firms' (newly issued) bonds or equities and one for government bonds. Accordingly, one part of savings \( S_{pr} \) is invested in bonds/equities issued by private firms

\[ S_{pr}^f = s_{pr}^f (Y - T + r_B B) \]  

(9),

another in government bonds

\[ S_{pr}^g = s_{pr}^g (Y - T + r_B B) \]  

(10),\(^7\)

where

\[ s_{pr} = s_{pr}^f + s_{pr}^g \]  

(11).

Thus, in this as in Solow's model bonds and equities are held directly by households, whereas in reality government bonds are held to a large extent by intermediaries (see for recent data e.g. Andritzky 2012 and Broner et al. 2014).

In the following, for brevity's sake \( T \) will be called taxes, and the corresponding rate \( \tau \) tax rate. Furthermore, as in Solow's original contribution we just stipulate consumption, saving, and investment functions without deriving them explicitly through a utility maximization exercise.\(^8\) In any case, implicitly we assume that households' instantaneous utility depends on both consumption of goods and wealth which is invested in firm equities or bonds and government bonds according to the respective saving rates.

Taxes depend on the tax rate \( \tau \) and gross household income \((Y + r_B B)\), which flows from private firms as labor and capital income and from government as interest payments on government bonds

\[ T = \tau (Y + r_B B) \]  

(12).

By postulating the saving functions (9) and (10), we assume that households' preferences are such that they want to diversify their investments by investing their savings both in private equities/bonds and

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\(^6\) In the following, for brevity's sake the term 'net' will be omitted in general. Just as in Solow's original contribution to the theory of economic growth (Solow 1956) we do not treat depreciation explicitly in the model. But we will consider depreciation when we come to the Maastricht criteria concerning the debt-to-GDP and the public-deficit-to-GDP ratio.

\(^7\) This assumption is in line with von Weizsäcker 2014.

\(^8\) See for a theoretical justification Akerlof's Presidential Address at the Annual Meeting of the American Economic Association 2007 (Akerlof 2007, especially pp 13ff.).
government bonds. Furthermore, we assume that private households just reinvest their financial means in firm and government bonds, whenever outstanding firm and government bonds become due. Alternatively we can assume that firms and governments issue perpetual bonds. Secondary markets for equities and government bonds are not explicitly modeled.

Savings that flow to private firms $S_{pr}^f$ are used to finance private firms’ net investments $I_{pr}$ in private real capital $K_{pr}$

$$I_{pr} = K_{pr} = S_{pr}^f$$  (13). 9

Savings that flow to government $S_{pr}^g$ are used to buy new government bonds $B^D$ which finance the government’s budget deficit $\dot{B}$:

$$\dot{B} = B^D = S_{pr}^g$$  (14).

Here and in the following a dot above a variable denotes the first derivative with respect to time. Eq. (14) has the following economic implication: the government supplies additional government bonds that private households demand according to their disposable income and risk preferences. Government budget deficits respond to the portfolio choices of private households just as firms’ investments, i.e. firms’ deficits. Both are determined by the demand of private households for new assets. Government budget deficits are an equilibrium phenomenon just as private firms’ deficits financed by issuing stocks or bonds. For the corresponding public-deficit-to-national-income ratio we get by simple division of eq. (14) by national income:

$$\frac{\dot{B}}{Y} = \frac{S_{pr}^g}{Y}$$  (15).

The government’s liquidity constraint is:

$$T + \dot{B} = C_g + I_g + \delta B$$  (16).

Excluding monetization of public debt, the government’s total revenue consists of taxes plus net increase in government debt $\dot{B}$. This revenue is used to finance public expenditures on consumption $C_g$ and investment $I_g$ plus interest payments on outstanding government debt $\delta B$.

The government’s primary deficit $D$ is defined as follows:

$$D = C_g + I_g - T$$  (17),

and the corresponding ratio of the primary deficit to national income $d$ as:

$$d = \frac{D}{Y} = \frac{C_g + I_g - T}{Y}$$  (18).

From eqs (14) and (17) we obtain:

$$\dot{B} = D + \delta B$$  (19).

With the public-debt-to-national-income ratio $b$

$$b = \frac{B}{Y}$$  (20).

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9 Here and in the following we use the following abbreviations: $\dot{x} = dx / dt$ and $\ddot{x} = \dot{x}/x$. 
we get from eqs (18) - (20):
\[ d = (B - r_B)b \] 
(21).

For \( b > 0 \) there is a primary deficit \( (d > 0) \) if the rate of change of public debt exceeds the real rate of interest on public debt, and there is a primary surplus if the real rate of interest on public debt exceeds the rate of change of public debt.\(^{10}\)

From eqs (19), (20), (14), (12), and (10) we can also obtain another expression for the primary deficit ratio, namely:
\[ d = s_{pr}^g (1 - \tau) - r_B \left(1 - s_{pr}^g (1 - \tau)\right)b \] 
(22).

Hence, the primary debt ratio does not only depend on households’ propensity to invest in government bonds and the tax rate, but on the ratio of government’s interest payments to national income as well, as the latter influence households’ disposable income.

The budget deficit can be used to finance government’s net investments in public real capital \( K_g \) or a part of them as well as to finance a part of public consumption or interest payments on outstanding government debt. We introduce the public-investment-to-budget-deficit ratio \( \lambda \) that is supposed to be set by the parliament when it decides on the budget:
\[ \lambda = \frac{l_B}{B} \] 
(23).

In case of \( \lambda = 1 \) the so-called Golden Rule of Public Finance is followed (see e.g. Bassetto and Sargent and 2006)\(^{11}\). For public capital accumulation we obtain:
\[ K_g = l_B = \lambda B; \quad \lambda > 0 \] 
(24).

From eqs (16) and (24) we can deduce:
\[ T = C_g + r_B B + (1 - 1/\lambda)l_B \] 
(25).

If the Golden Rule of Public Finance is followed, taxes on national income and interest payments on public debt just serve to finance public consumption and interest payments on public debt. From eqs (12) and (25) we get for the share of public consumption in national income:
\[ \frac{C_g}{Y} = \tau - (1 - \tau)r_B \frac{B}{Y} - (1 - 1/\lambda) \frac{l_B}{Y} \] 
(26),

and from eq. (12) for the share of taxes in national income:
\[ \frac{T}{Y} = \tau(1 + r_B \frac{B}{Y}) \] 
(27).

From eqs (4) - (6), (8) - (10), (13), (20), (24) and (27) the following expression can be derived for the share of public consumption in national income:
\[ \frac{C_g}{Y} = 1 - (1 - \lambda) \frac{s_{pr}^g}{(1 - \tau)(1 + r_B b)} \] 
(28).\(^{12}\)

\(^{10}\) Contrary to Gärtner 2009 and de Grauwe 2014 the ratio of the primary deficit to national income is endogenous.

\(^{11}\) See in this context also Buiter 2001, Blanchard and Giavazzi 2004 and Sachverständigenrat 2007 pp. 3ff.

\(^{12}\) It should be noted that eqs (26) and (28) are not independent from each other.
The share of public consumption in national income is an endogenous variable in this model which is
determined by the government’s liquidity constraint, and hence (i) by the public-deficit-to-national-
income ratio which in turn is determined by households’ propensity to invest in government bonds, (ii) by
the share of taxes in national income which in turn is determined by the tax rate and the share of inter-
est payments on public debt in national income, and (iii) by the share of public investment in national
income which in turn is determined by the public-investment-to-budget-deficit ratio and, again, by the
public-deficit-to-national-income ratio.

As in Solow’s growth model we assume perfect competition in the markets for goods and services, labor
and private capital. The price level is assumed to be unity. As all incomes flow to households only pri-
vate households pay income tax. Hence, the rental price of private capital $R$ equals the marginal
productivity of private capital and the wage rate equals the marginal productivity of labor:

$$R = \gamma K_{pr} = \beta A^{\alpha} L^{\alpha} K_{pr}^{1-\gamma} L^{1-\alpha-\beta} = \beta \frac{Y}{K_{pr}} \quad (29),$$

$$w = \gamma L = \alpha A^{\alpha} L^{\alpha-1} K_{pr}^{\beta} L^{1-\alpha-\beta} = \alpha \frac{Y}{L} \quad (30).$$

Furthermore, we assume that the use of public capital is free of charge. This leads to profits $(\Pi)$ even
with perfect competition:

$$(\Pi) = (1 - \alpha - \beta)Y \quad (31).$$

For the respective rate of profit $r_{\Pi}$

$$r_{\Pi} = \frac{\Pi}{K_{pr}} \quad (32),$$

we can compute from eq. (29):

$$r_{\Pi} = (1 - \alpha - \beta) \frac{Y}{K_{pr}} \quad (33),$$

and hence, for the overall rate of return on private capital $r_K$

$$r_K = R + r_{\Pi} \quad (34).$$

From eqs (29), (33), and (34) follows:

$$r_K = (1 - \alpha) \frac{Y}{K_{pr}} \quad (35).$$

Even assuming perfect markets, we allow that the rate of return on private capital may exceed the one
on government bonds due to risk and liquidity considerations of private households. This implies that
households consider an investment in government bonds less risky and more liquid than an investment
in private enterprises. Private households’ risk and liquidity considerations are taken into account by the
exogenously given safety discount factor $\sigma$. Hence, we postulate

$$r_B = (1 - \sigma) r_K; \quad 0 \leq \sigma < 1 \quad (36).$$

By assumption, the safety discount factor does not exceed unity. Hence, we consider it a disequilibrium
phenomenon when even the nominal rate of interest on government bonds is negative like the German
Bund during certain periods of the Euro crisis.

Finally, we define private and public capital-labor ratios in efficiency units:

$$\kappa_{pr} = \frac{K_{pr}}{AL} \quad (37).$$
\[ \kappa_g = \frac{K_g}{AL} \]  
\hspace{1cm} (38),

as well as net domestic product per capita in efficiency units:

\[ y = \frac{Y}{AL} \]  
\hspace{1cm} (39).

The production function can be rewritten by using eqs (39), (37) and (38):

\[ y = \frac{Y}{AL} = \gamma \frac{(AL)^\alpha}{(AL)^\alpha} \frac{K_{pr}^\beta}{(AL)^{1-\alpha-\beta}} = \gamma \kappa_{pr}^\beta \kappa_g^{1-\alpha-\beta} = y \left( \kappa_{pr}, \kappa_g \right) \]  
\hspace{1cm} (40).

From eq. (24) we obtain

\[ K_g = \lambda B \]  
\hspace{1cm} (41)

if \( \lambda \) remains unchanged over time, and hence according to eq. (20) for the debt ratio \( b \):

\[ b = \frac{B}{Y} = \frac{B/AL}{Y/AL} = \frac{K_g}{\lambda y} = \frac{\kappa_g}{\lambda \gamma k_{pr}^\beta \kappa_g^{1-\alpha-\beta}} = \lambda^{-1} \gamma^{-1} \kappa_{pr}^\beta \kappa_g^{1-\alpha-\beta} \]  
\hspace{1cm} (42).

For the interest rate on public debt we can derive from eqs (35) and (36)

\[ r_B = (1 - \sigma)(1 - \alpha) \frac{Y}{K_{pr}} = \gamma (1 - \sigma)(1 - \alpha) \frac{\kappa_{pr}^{\beta-1} \kappa_g^{1-\alpha-\beta}}{\kappa_{pr}} \]  
\hspace{1cm} (43),

and hence with eqs (43) and (42) for the share of government’s interest payments in national income:

\[ r_B b = \lambda^{-1} (1 - \sigma)(1 - \alpha) \frac{\kappa_g}{\kappa_{pr}} \]  
\hspace{1cm} (44),

and hence with eq. (27) for the ratio of households’ disposable income to national income:

\[ \frac{Y - T + r_B B}{Y} = 1 - \frac{T}{Y} + r_B \frac{B}{Y} = 1 - \tau + (1 - \tau)r_B b = (1 - \tau) \left( 1 + \lambda^{-1}(1 - \sigma)(1 - \alpha) \frac{\kappa_g}{\kappa_{pr}} \right) \]  
\hspace{1cm} (45).

From eqs (45), (9), and (10) we can deduce the shares of households’ investment in private and public bonds in national income:

\[ \frac{S_{pr}^f}{Y} = \frac{s_{pr}^f}{Y}(1 - \tau) \left( 1 + \lambda^{-1}(1 - \sigma)(1 - \alpha) \frac{\kappa_g}{\kappa_{pr}} \right) \]  
\hspace{1cm} (46),

\[ \frac{S_{pr}^g}{Y} = \frac{s_{pr}^g}{Y}(1 - \tau) \left( 1 + \lambda^{-1}(1 - \sigma)(1 - \alpha) \frac{\kappa_g}{\kappa_{pr}} \right) \]  
\hspace{1cm} (47).

Finally, from eqs (35) and (40) we derive for the rate of return on private capital:

\[ r_K = (1 - \alpha) \frac{Y}{K_{pr}} = \gamma (1 - \alpha) \frac{\kappa_{pr}^{\beta-1} \kappa_g^{1-\alpha-\beta}}{\kappa_{pr}} \]  
\hspace{1cm} (48).
3. Steady State and Comparative Dynamics

In this section we deal with growth dynamics. Here we derive the system of differential equations for the private and the public capital-labor ratio in efficiency units\(^{13}\) and the steady state. With \(\dot{^\gamma}\) denoting the percentage rate of change, we get from eq. (13) and eqs (37) - (39), and (45):

\[
\dot{K}_{pr}^g = s_{pr}^f (1 - \tau) (\kappa_{pr}^{pr} - 1) \kappa_{pr}^{-1} \kappa_{g}^{1-\alpha-\beta} + \lambda^{-1}(1-\sigma)(1-\alpha)\kappa_{pr}^{pr} \kappa_g^{2-\alpha-\beta})
\]  

(49).

From eq. (24) we get with eq. (10) and eqs (37) - (39), and (45):

\[
\dot{K}_g^g = \lambda s_{pr}^g (1 - \tau) \gamma (\kappa_{pr}^{pr} \kappa_g^{1-\alpha-\beta} + \lambda^{-1}(1-\sigma)(1-\alpha)\kappa_{pr}^{pr} \kappa_g^{2-\alpha-\beta})
\]  

(50).

From eqs (37), (2) and (3) we can derive the percentage rate of change of the private capital-labor ratio in efficiency units:

\[
\dot{K}_{pr}^g = \dot{K}_{pr} - \delta - n
\]  

(51),

and from eqs (38), (2) and (3) the percentage rate of change of the public capital-labor ratio in efficiency units:

\[
\dot{K}_g^g = \dot{K}_g - \delta - n
\]  

(52).

We obtain the system of differential equations by inserting eq. (49) into eq. (51):

\[
\dot{K}_{pr}^g = s_{pr}^f (1 - \tau) \gamma (\kappa_{pr}^{pr} \kappa_g^{1-\alpha-\beta} + \lambda^{-1}(1-\sigma)(1-\alpha)\kappa_{pr}^{pr} \kappa_g^{2-\alpha-\beta}) - (\delta + n)\kappa_{pr}^f
\]  

(53),

and eq. (50) into eq. (52):

\[
\dot{K}_g^g = \lambda s_{pr}^g (1 - \tau) \gamma (\kappa_{pr}^{pr} \kappa_g^{1-\alpha-\beta} + \lambda^{-1}(1-\sigma)(1-\alpha)\kappa_{pr}^{pr} \kappa_g^{2-\alpha-\beta}) - (\delta + n)\kappa_g^f
\]  

(54).

The steady state \((\kappa_{pr}^*, \kappa_g^*)\) is derived by setting both eqs (53) and (54) equal to zero. By dividing the two resulting equations, we can see that the following relationship holds in the steady state:

\[
\kappa_g^* = \frac{s_{pr}^g}{s_{pr}^f} \kappa_{pr}^*
\]  

(55).

We define the ratio of the propensities to save:

\[
s_{pr}^{fg} = \frac{s_{pr}^f}{s_{pr}^g}
\]  

(56).

By inserting eq. (55) into eqs (46) and (47) we can derive the steady state values of the shares of households' investment in private and public bonds in national income:

\[
\frac{\partial s_{pr}^f}{\partial \tau} < 0, \frac{\partial s_{pr}^f}{\partial \sigma} < 0, \frac{\partial s_{pr}^f}{\partial \alpha} < 0, \frac{\partial s_{pr}^f}{\partial \kappa_{pr}} > 0, \frac{\partial s_{pr}^f}{\partial \kappa_g^f} > 0
\]  

(57)

and

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\(^{13}\) For simplicity’s sake, in the following text we will often omit ‘in efficiency units’.
\begin{align*}
\ddot{s}_g^{pr} &= s_g^{pr}(1-\tau) \left( 1 + (1-\sigma)(1-\alpha) \left( s_{g}^{pr} \right)^{-1} \right) ; \\
\frac{\partial s_g^{pr}}{\partial \tau} &< 0, \frac{\partial s_g^{pr}}{\partial \sigma} < 0, \frac{\partial s_g^{pr}}{\partial \alpha} < 0, \frac{\partial s_f^{pr}}{\partial s_{pr}} < 0, \frac{\partial s_g^{pr}}{\partial s_{pr}} > 0 
\end{align*}
(58).

Finally, in the steady state, we can rewrite eqs (53) and (54) in the following way:

\[ \ddot{s}_f^{pr} y \left( \kappa_{pr}^*, \kappa_{g}^* \right) = (\delta + n) \kappa_{pr}^* \]
(59),

and

\[ \lambda \ddot{s}_g^{pr} y \left( \kappa_{pr}^*, \kappa_{g}^* \right) = (\delta + n) \kappa_{g}^* \]
(60).

Eqs (59) and (60) are the equivalents to the well-known equilibrium condition in the Solow model: \( s_{pr}^f y(\kappa_{pr}^*) = (\delta + n)\kappa_{pr}^* \). Households' propensity to save and invest in private capital with respect to national income \( s_{pr}^f \) decreases with the tax rate and the safety discount factor, because in both cases households' disposable income shrinks. The same holds for the households' propensity to save and invest in government bonds with respect to national income \( s_g^{pr} \) which, according to eq. (15), equals the steady state public-deficit-to-national-income ratio:

\[ \left( \frac{B}{Y} \right)^* = s_g^{pr} = s_g^{pr}(1-\tau) \left( 1 + (1-\sigma)(1-\alpha) \left( s_{pr}^f \right)^{-1} \right) \]
(61).

From eq. (61) we see that the steady state public deficit ratio is constant, just as in the original Solow growth model the ratio of the change of capital to national income is equal to the savings rate which is constant \( (K_{pr} / Y = s_{pr}^f) \).14

We obtain the steady state values of private and public capital per capita by inserting eq. (55) into eq. (53):15

\[ \kappa_g^* = \left[ \frac{\gamma \lambda^1-\beta \left( s_{g}^f \right)^{\beta} s_{g}^g}{(\delta + n)} \right] \]
(62),

\[ \frac{\partial \kappa_g^*}{\partial (\delta + n)} < 0, \frac{\partial \kappa_g^*}{\partial \tau} < 0, \frac{\partial \kappa_g^*}{\partial \sigma} < 0, \frac{\partial \kappa_g^*}{\partial \alpha} < 0, \frac{\partial \kappa_g^*}{\partial s_{pr}} > 0, \frac{\partial \kappa_g^*}{\partial s_{g}^{pr}} > 0, \frac{\partial \kappa_g^*}{\partial \lambda} > 0 
\]

and finally, again using eq. (55):

14 With a more general production function than the Cobb-Douglas production function the sum of the production elasticities of private and public capital \( 1 - \alpha \) and hence the public deficit ratio would not be exogenously given.
15 As in the original Solow model there is a second steady state, namely \( \kappa_g = \kappa_{pr} = 0 \). This steady state is an unstable node.
The parameter values in the various figures are based on rough estimates for Germany.
By inserting eqs (62) and (63) into eq. (42), and using eq. (58), we get for the steady state value of the ratio of public debt to national income:

\[ b^* = \frac{s_{pr}^g (1 - \tau)}{\delta + n} (1 + (1 - \sigma)(1 - \alpha) \left( \frac{f_{pr}}{s_{pr}} \right)^{-1}) = \frac{s_{pr}^g}{\delta + n}; \]

\[ \frac{\partial b^*}{\partial (\delta + n)} < 0, \frac{\partial b^*}{\partial \tau} < 0, \frac{\partial b^*}{\partial \sigma} < 0, \frac{\partial b^*}{\partial \alpha} < 0, \frac{\partial b^*}{\partial s_{pr}^g} < 0, \frac{\partial b^*}{\partial s_{pr}^g} > 0, \frac{\partial b^*}{\partial \lambda} = 0. \] (66).

Eq. (66) shows the well-known result that the steady state value of the debt ratio is equal to the steady state public deficit ratio divided by the steady state growth rate of national income. Under the additional assumptions \( s_{pr}^g > 0 \) and \( \delta + n > 0 \) the steady state debt ratio \( b^* \) is also positive. According to eq. (66) the steady state debt ratio is the larger, the larger the households’ propensity to save and invest in government bonds \( s_{pr}^g \) and the larger the sum of the partial production elasticities of private and public capital \( (1 - \alpha) \). The steady state debt ratio is the smaller, the larger the equilibrium growth rate \( (\delta + n) \), the tax rate \( (\tau) \), the safety discount factor \( (\sigma) \), and the households’ propensity to save and invest in shares of private firms \( s_{pr}^f \).\footnote{The signs of the impacts of variations of the growth rate and the tax rate on the steady state public debt ratio are the same as in the simple public debt dynamics model of Gärtner 2009 in the case of a positive stable equilibrium public debt ratio.} What may be surprising at first sight is the fact that the steady state debt ratio does not depend on the public-investment-to-budget-deficit ratio \( \lambda \). But this fact is in line with the government budget deficit being determined by private households’ demand for government bonds as an asset for their savings. Government can influence the steady state public deficit ratio and hence the steady state debt ratio only by varying the tax rate, thus influencing disposable income and hence households’ demand for newly issued government bonds.

Figure 2 shows how the steady state public deficit ratio varies mainly with households’ propensity to invest in government bonds. The respective slope is close to unity. It is the larger the smaller is the tax rate. The Maastricht criterion of 3 % refers to the debt-to-GDP ratio. The corresponding criterion for the public-deficit-to-NDP ratio \( b \) would be about \( 15 - 20 \% \) higher because of \( \frac{\dot{B}}{Y} = \left( \frac{\dot{B}}{Y} \right)^{Gr} \left( \frac{Y^{Gr}}{Y} \right) \), where \( Y^{Gr} \) denotes GDP and \( \left( \frac{\dot{B}}{Y} \right)^{Gr} \) the public-deficit-to-GDP ratio. The same reasoning applies to
the debt ratio. Yet, for simplicity’s sake, in the figures we neglect the distinction between public-deficit-to-NDP ratio and public-deficit-to-GDP ratio.

By multiplying eqs (64) and (66) or by dividing eq. (62) by $\lambda$, we can derive the steady state value of public debt per capita:

$$y^*_b = \gamma^{1/\alpha} \lambda^{(1-\alpha+\beta)/\alpha} (\delta + n)^{-1/\alpha} \left(\frac{s^g_{pr}}{r}\right)^{1/\alpha}$$

Even if, as pointed out above, the steady state value of the ratio of public debt to national income remains unaffected by changes in $\lambda$, this does not hold for the steady state value of public debt per capita. This increases with $\lambda$ as does per capita income.

By inserting eq. (55) into eq. (44) we derive for the steady state value of the ratio of public interest payments to national income:

$$r^*_b = \frac{s^g_{pr}(1-\sigma)(1-\alpha)}{s_{pr}};$$

For the steady state value of the ratio of public interest payments to national income to be smaller than unity it is sufficient that households invest at least as much in private firms as in government bonds. According to eq. (68) there is a negative relationship between the steady state values of the interest rate on government bonds and the public debt ratio. The lower the public debt ratio, the higher is the rate of interest on government bonds, given the safety discount factor, labor’s partial production elasticity, and the ratio of saving rates. Remarkably, government cannot influence the ratio of public interest payments to national income by changing the tax rate. In fact, in the presented model the steady state value of the ratio of public interest payments to national income is determined by preferences and risk considerations of private households and by technology, not by government.\(^{18}\)

If we divide eq. (68) by eq. (66) we finally obtain the steady state value of the interest rate on public debt:

$$r^*_B = \frac{(\delta + n)(1-\sigma)(1-\alpha)}{(1-\tau)(s_{pr} + s^g_{pr})(1-\sigma)(1-\alpha)} \left(\frac{s^g_{pr}}{s_{pr}}\right);$$

Eq. (69) is the analogous expression of the steady state value of the real rental price of (private) capital in the original Solow growth model, namely $r^*_K = (\delta + n)(1-\alpha) / s^g_{pr}$.\(^{19}\) According to eq. (69) the steady

\(^{18}\) This result depends on the production function being of the Cobb-Douglas-type.

\(^{19}\) From this equation follows that the real rate of interest in Solow’s model exceeds the steady state growth rate as long as the saving rate exceeds the partial production elasticity of capital $1-\alpha$ . An analogous result was derived by Pasinetti 1962, where he shows that the steady state real rate of interest equals the ratio of the steady state growth rate to the saving rate of capitalists under the additional condition that the distribution of wealth between (saving) workers and capitalists remains
The state rate of interest on government bonds decreases with an increase in the propensities to save and the safety discount factor ($\sigma$) as well as with a decrease in the equilibrium growth rate ($\delta + n$) and the tax rate ($\tau$). Neither the steady state value of the ratio of public interest payments to national income nor the steady state rate of interest on government bonds depend on the ratio of public investment to the budget deficit $\lambda$.

Figure 3 shows that the steady state growth rate exceeds the steady state rate of interest only for rather high values of the safety discount factor.

By using eqs (18), (24), (27), (28), (66) - (69) the equilibrium value of the ratio of the primary deficit to national income $d$ can be computed as:

$$
\left( s_{pr}^f (1-\tau) - (1-\sigma)(1-\alpha) \left( 1 - s_{pr}^g (1-\tau) \right) \right) \left( s_{pr}^f \right)^{-1} ;
$$

$$
\frac{\partial d^*}{\partial s_{pr}^f} > 0, \frac{\partial d^*}{\partial s_{pr}^g} < 0, \frac{\partial d^*}{\partial \tau} < 0, \frac{\partial d^*}{\partial \sigma} > 0, \frac{\partial d^*}{\partial \alpha} > 0, \frac{\partial d^*}{\partial \lambda} = 0
$$

It can be positive or negative or zero. From eqs (21) and (65) we directly get the following relationship between the steady state values of the primary deficit ratio, the debt ratio, and the difference between the growth rate of real national income and the real rate of interest on government bonds:

$$
d^* = (\hat{Y}^* - \hat{f}_B) b^*
$$

For $b^* > 0$ we can derive:

$$
d^* \leq 0 \leftrightarrow \hat{Y}^* \geq \hat{f}_B
$$

Hence, for $b^* > 0$, in the steady state there is a primary deficit if the growth rate of real national income exceeds the real rate of interest on public debt, and there is a primary surplus if the growth rate of real national income is smaller than the real rate of interest on public debt.

Figure 4 shows that the steady state values of the primary deficit ratio and of the difference between the growth rate of real national income and the real interest rate on public debt are more sensitive with respect to changes in the safety discount factor than to changes in the tax rate. If the government increases the tax rate in order to turn the steady state value of a primary deficit into a primary surplus this unchanged. This clearly shows that Piketty’s claim that the wealth distribution becomes more and more unequal if the interest rate exceeds the growth rate does not apply to models with representative agents (see Piketty 2014).
increases the steady state value of the real interest rate on public debt in such a way that the difference between the steady state values of the growth rate of real national income and the real interest rate on public debt turns negative, too. In this sense, for given values of the other parameters, government cannot only influence the sign of the primary deficit, but the sign of the difference between the growth rate of real national income and the real rate of interest as well.

The reason for (72) to hold is the government’s liquidity constraint (see eq. (19)). If $\hat{B} \hat{r} Y > 0$ in the steady state then government’s interest payments exceed the budget deficit. According to eq. (19) this gap has to be filled by a primary surplus:

$$f_B \geq \hat{Y}^* \iff f_B \hat{b}^* = f_B \hat{B} Y \geq \hat{Y}^* b^* = \hat{B} Y \iff \hat{D}^* Y = \hat{d}^* \leq 0 \quad (73).$$

If a debt brake is introduced as in Germany (see Feld 2010), where the debt-brake-public-deficit-to-GDP ratio is restricted to 0.35 % in the constitution:

$$\left( \frac{\hat{B}}{Y} \right)_\text{GER} = \left( \frac{\hat{s}_g}{\hat{s}_\text{pr}} \right)_\text{GER} = 0.0035 \frac{Y^g_{\text{GER}}}{Y_{\text{GER}}} \quad (74),$$

with DB referring to debt brake and GER to Germany, the corresponding debt-brake-tax rate $\tau_{\text{DB}}$ can be computed from eq. (58):

$$\tau_{\text{DB}} = 1 - \frac{\left( \frac{\hat{s}_g}{\hat{s}_\text{pr}} \right)_{\text{DB}}}{\hat{s}_\text{pr} \left( 1 + (1 - \sigma) (1 - \alpha) \left( \frac{\hat{f}_g}{\hat{s}_\text{pr}} \right)^{-1} \right)} \quad (75).$$

From eq. (75) we can see that the debt brake tax rate increases with households’ propensity to invest in government bonds. Thus, in order to satisfy the debt brake, the government has to increase the tax rate when private households increase their propensity to invest in government bonds, thus directly acting against private households’ preferences. The debt brake tax rate decreases with an increase of the safety discount factor, because an increase in the safety discount factor reduces households’ interest income from government bonds and this lowers households’ propensity to save and invest in government bonds with respect to national income.

By inserting the debt-brake-public-deficit ratio into eq. (66) we can compute the corresponding steady state debt-brake-public-debt ratio:
According to eq. (76) a decrease of the public deficit ratio due to the introduction of a debt brake does not only reduce the steady state public debt ratio, but according to eq. (64) also per capita income unless government increases the public-investment-to-budget-deficit ratio sufficiently in order to counterbalance the adverse impact. By differentiating eqs (64) and (58) we can compute the percentage change of the public-investment-to-budget-deficit ratio $\lambda$ which is necessary in order to keep national income per capita unchanged if the tax rate is increased in order to reduce the public deficit ratio:

$$\frac{d\lambda}{\lambda} = -\frac{(1-\alpha)}{(1-\alpha-\beta)} \frac{ds^g_{pr}}{\delta^g_{pr}} = \frac{(1-\alpha)}{(1-\alpha-\beta)(1-\tau)} d\tau$$ (77).

### 4. Sustainability of Public Debt Dynamics

We now turn to the question posed in the introduction, namely: Is it really true that in a growing economy a constant debt-to-GDP ratio can only be stable if the real rate of interest on government debt is lower than the growth rate of real GDP which in turn leads to a violation of the government’s solvency constraint? To answer this question we will first describe Romer’s approach (Romer 2012) and how it fits into this model. Then we describe the approach by Gärtner 2009 and de Grauwe 2014 (G2), and finally we compare their approach with the one presented here.

Following Romer 2012, p. 586 we write the government’s solvency constraint in the following form:

$$\lim_{t \to \infty} PVB(t) = \lim_{t \to \infty} B(t) e^{-R_B(t)} \leq 0$$ (78).

With

$$R_B(t) = \int_{\psi=0}^{t} r(\psi)d\psi$$ (79),

where $PVB$ denotes the present value of public debt in $t = 0$ and $R_B(t)$ the compound interest on government bonds from $t = 0$ until $t$. As public debt is positive as long as households’ propensity to invest in government bonds is positive, the present value of public debt has to converge to zero in order to satisfy the government’s solvency constraint in our model. Hence, the rate of change of $PVB$ has to be negative when time goes to infinity:

$$\lim_{t \to \infty} (PVB(t)) = \lim_{t \to \infty} \hat{B}(t) - \lim_{t \to \infty} r_B(t) < 0$$ (80).

Obviously, if the model economy is not in the steady state right from the start, the steady state values are only approached if the steady state is asymptotically stable. Only under this condition holds:

$$\lim_{t \to \infty} (PVB(t)) = \hat{B}^* - r_B^* = (PVB)^* < 0$$ (81).

---

20 If we assume an average growth rate of nominal GDP of about 3% for Germany the resulting long run debt brake public debt ratio is about 10%.

21 This formulation is more general than the one used by Blanchard et al 1990, p. 12, who assume a constant rate of growth of real GDP and a constant real rate of interest.
Hence, the stability of the steady state also matters for the answer to the question of whether the government’s solvency constraint is satisfied or not. In the steady state the rate of change of public debt equals the steady state growth rate of national income \((\delta + n)\) according to eq. (65). Thus from (81) follows the minimum threshold of the steady state rate of interest on government bonds \(r_{BMin}^{PVB}\) that has to be exceeded in order to satisfy the solvency constraint:

\[ r_{BMin}^{PVB} = \delta + n \]  

(82)

Taking into account eq. (69) condition (81) can be rewritten as:

\[ \left(\frac{s_{pr}^g f(1-\sigma)(1-\alpha)}{s_{pr}^g f + s_{pr}^g (1-\sigma)(1-\alpha)}\right) < 0 \]

(83).

According to eq. (83), for \((\delta + n) > 0\) the sign of the steady state rate of change of the present value of public debt does not depend on the size of the steady state growth rate of national income \((\delta + n)\), as the latter influences the steady state values of both the rate of change of public debt and the rate of interest on government bonds in a multiplicative manner. Instead, the sign of the steady state rate of change of the present value of public debt depends on all the other parameters that influence the steady state rate of interest on government bonds alone (see the fraction in eq. (83) and taking into account eq. (69)). Considering the partial derivatives of the steady state rate of change of the present value of public debt, it is certainly not surprising that a higher tax rate helps to satisfy the government’s solvency constraint. From eq. (83) and eq. (61) follows as the minimum threshold of the tax rate \(\tau_{Min}^{PVB}\) that has to be exceeded for the government’s solvency constraint to be satisfied:

\[ \tau > \tau_{Min}^{PVB} = 1 - \frac{(1-\sigma)(1-\alpha)}{s_{pr}^f + s_{pr}^g (1-\sigma)(1-\alpha)} ; \quad \frac{\partial \tau^{PVB}}{\partial \sigma} > 0, \quad \frac{\partial \tau^{PVB}}{\partial \alpha} > 0, \quad \frac{\partial \tau^{PVB}}{\partial s_{pr}^f} > 0, \quad \frac{\partial \tau^{PVB}}{\partial s_{pr}^g} > 0 \]  

(84).

If the tax rate exceeds the minimum threshold there is a primary surplus in the steady state according to (72). Hence, in order to fulfill the solvency constraint it is sufficient for the government to realize a primary surplus in the steady state, as long as the steady state is stable. The minimum threshold of the tax rate increases with increases in the safety discount factor, the production elasticity of labor, and households’ saving rates (see Figure 5).

Alternatively, for a given tax rate, from eq. (83) and eq. (61) follows as maximum threshold of the safety discount factor \(\sigma_{Max}^{PVB}\) that is not to be exceeded for the government’s solvency constraint to be satisfied:

\[ \sigma < \sigma_{Max}^{PVB} = 1 - \frac{s_{pr}^f (1-\tau)}{(1-\alpha)(1-s_{pr}^g (1-\tau))} ; \quad \frac{\partial \sigma_{Max}^{PVB}}{\partial \tau} > 0, \quad \frac{\partial \sigma_{Max}^{PVB}}{\partial \alpha} < 0, \quad \frac{\partial \sigma_{Max}^{PVB}}{\partial s_{pr}^f} < 0, \quad \frac{\partial \sigma_{Max}^{PVB}}{\partial s_{pr}^g} < 0 \]  

(85).

22 Since the outbreak of the Great Recession in many countries the central banks have lowered the level of interest rates at which banks can refinance themselves. In the majority of these countries this has led to a decline of the rate of interest for government bonds. As a first approximation, such a decline can be modeled by an increase in the safety discount factor.
At first sight it may seem counterintuitive that a higher safety discount factor and hence a lower steady state rate of interest rate on government bonds, a lower steady state ratio of government’s interest payments to national income, and a lower steady state level of public debt makes it more difficult to satisfy the government’s solvency constraint in the long run. But this result is due to the fact that the steady state growth rate of government debt is not affected by a change in the safety discount factor, whereas the steady state rate of interest on government bonds is lowered. Figure 5 shows how strongly the minimum threshold of the tax rate increases with the safety discount factor. For small values of the safety discount factor the minimum threshold of the tax rate is negative and hence smaller than the actual tax rate. But for large values of the safety discount factor a further increase may require a substantial increase of the tax rate in order to satisfy the government’s solvency constraint.

Figure 5: Minimum threshold for the tax rate as a function of the propensity to invest in government bonds and the safety discount factor

\[ \gamma = \alpha = 0.7; \beta = 0.2; n = 0; \delta = 0.02; \lambda = 1 \]

For \( (\delta + n) > 0 \) inequality (83) is satisfied if

\[
\left( \frac{B}{Y} \right)^* = s_{pr}^g < \frac{s_{pr}^g (1 - \alpha)(1 - \alpha)}{s_{pr}^f} = \left( \frac{B}{Y} \right)^* = \frac{B}{Y}
\]

i.e., if the steady state public-deficit-to-national-income ratio is smaller than the steady state government-interest-payments-to-national-income ratio. According to inequality (86) the steady state government-interest-payments-to-national-income ratio is the upper threshold of the steady state public-deficit-to-national-income ratio \( \left( \frac{B}{Y} \right) \) that is not to be reached if the government’s solvency constraint is to be satisfied. This upper threshold is the lower, the lower is the propensity to invest in public bonds compared to the propensity to invest in private bonds/shares, the lower is capital’s share in national income, and the higher is the safety discount factor.

Since the outbreak of the Great Recession in many countries the central banks have lowered the level of interest rates at which banks can refinance themselves. In the majority of these countries this has led to a decline of the rate of interest for government bonds. As a first approximation, such a decline can be modeled by an increase in the safety discount factor.
But, as Figure 6 shows, by setting a tax rate that exceeds the minimum tax rate according to inequality (84) the government has an instrument to make sure that its solvency constraint is satisfied, even if the safety discount is large. For the corresponding upper threshold of the steady state debt ratio follows from inequality (86):

$$b^* = \frac{s_{pr}^b}{\delta + n} < \frac{s_{pr}^b (1 - \sigma)(1 - \alpha)}{s_{pr}^b (\delta + n)} = \frac{\sigma b^*}{\delta + n} = b^*$$

(87).

From (73), (81), (86), and (87) follows

$$r_B^* \geq \frac{\hat{Y}}{Y} \Leftrightarrow r_B^* b^* = \left(\frac{\hat{B}}{Y}\right)^* \Leftrightarrow 0 \leq d^* \Leftrightarrow 0 \leq (PVB)^* \Leftrightarrow b^* \geq b^*$$

(88)

From eqs (80) and (83) we observe that satisfying the government’s solvency constraint as a condition for the sustainability of public debt requires the steady state rate of interest on government bonds to exceed the steady state growth rate of national income. But, according to Gärtner 2009 p. 395f and de Grauwe 2014 annex (G2), this condition leads to instability of the steady state debt ratio and hence non-sustainability of public debt. Thus let us have a closer look at their approach.

By differentiating the debt ratio with respect to time (see eq. (42)), we can compute

$$\dot{b} = \frac{\hat{B}Y - \hat{Y}B}{Y^2} = \frac{\hat{B}}{Y} - \dot{Y}b$$

(89).

By inserting eq. (16) into eq. (89) and taking into account the definition of the primary budget deficit the following differential equation for the debt-to-national-income ratio can be derived:

$$\dot{b} = \frac{Cg + lg - T}{Y} - (\hat{Y} - r_B)b = d - (\hat{Y} - r_B)b$$

(90),

from which – by setting $\dot{b} = 0$ – the equilibrium value of $b$ can be directly computed as:

$$b^* = \frac{d}{\hat{Y} - r_B}$$

(91),

---

24 Assuming the labor supply to depend on the tax rate might change this conclusion.

25 See in this context Greenlaw et al. 2013, too.

26 de Grauwe calls this the necessary condition for maintaining solvency (de Grauwe 2014). Hence, his use of the term ‘solvency’ is different from mine.
which is just the same equation as eq. (71) if $d$, $\hat{Y}$, and $r_B$ are equal to their steady state values. In principle, $d$, $\hat{Y}$, and $r_B$ could be functions of $b$. Hence, by linearizing eq. (90) around the equilibrium by means of a Taylor series of the first order we get:

$$b^T = d(b^*) - \left(\hat{Y}(b^*) - r_B(b^*)\right)b + d'_b(b^*)(b - b^*) - b^* \left(\hat{\dot{Y}}_b(b^*) - (r_B)_{b'}(b^*)\right)(b - b^*)$$  \hspace{1cm} (92),

where $T$ refers to Taylor and $d(b^*) = d^*$, $\hat{Y}(b^*) = \hat{Y}^*$, $r_B(b^*) = r_B^*$. G2 treat $d$, $\hat{Y}$, and $r_B$ as exogenous parameters. Hence they (implicitly) assume:

$$d'_b(b^*) = \hat{\dot{Y}}_b(b^*) = (r_B)_{b'}(b^*) = 0$$  \hspace{1cm} (93).

Accordingly, G2’s (linear) differential equation for the debt ratio can be written as follows:

$$\dot{b}^G = d^* - (\hat{Y}^* - r_B^*)b^G$$  \hspace{1cm} (94),

where $G$ refers to G2. As the growth rate and the rate of interest on government bonds do not directly depend on the debt ratio, we maintain the simplifying assumption $\dot{\hat{Y}}_b(b^*) = (r_B)_{b'}(b^*) = 0$ as a first approximation, but not $d'_b(b^*) = 0$, as the primary deficit ratio directly depends on the debt ratio according to eq. (22), hence:

$$d'_b(b^*) = -\left(1 - s^G pr(1 - \tau)\right)f^*_B$$  \hspace{1cm} (95).

By inspecting eq. (94) we see that the differential equation stipulated by G2 is linear, and intersects the vertical axis ($b^G$) at $d^*$ and the horizontal axis ($b^G$) at $b^* = b^{G*} > 0$. If $d^* < 0$ and hence $\hat{Y}^* < r_B^*$ the resulting differential equation $\dot{b}^G$ is positively sloped and, thus, the equilibrium seems to be unstable even if it is stable, as will be shown in the following.

In the model presented all the markets are assumed to be in equilibrium even outside the steady state. Hence, we are dealing with stability in the restricted sense of equilibrium dynamics, not disequilibrium dynamics (Hahn and Matthews 1964, p.782 and pp. 804 ff.). By linearizing the differential equations (53) and (54) in the relevant steady state we can compute the discriminant $\Delta$:

$$\Delta = (\delta + n)^2(1 - \alpha)^2 > 0 \quad \forall \quad (\delta + n) \neq 0; \quad \alpha \neq 1$$  \hspace{1cm} (96).

Hence, we have two real roots. For the trace $tr$ of the Jacobian matrix we get:

$$tr = -(\delta + n)(1 + \alpha) < 0 \quad \forall \quad (\delta + n) > 0; \quad \alpha > 0$$  \hspace{1cm} (97),

and for the determinant $det$ of the Jacobian matrix:

$$det = (\delta + n)^2\alpha > 0 \quad \forall \quad (\delta + n) \neq 0; \quad \alpha > 0$$  \hspace{1cm} (98).

Because of eqs (96) - (98) the relevant steady state is a stable node (see Gandolfo 2010, p.358), as long as $(\delta + n) > 0$ and $1 > \alpha > 0$.\hspace{1cm}27 It is interesting to note that the stability of the economically relevant steady state just depends on the sum of the growth rates of technological progress and labor supply as well as on the partial production elasticity of labor in efficiency units, not on households’ propensities to save, the tax rate, the safety discount factor or the ratio of public investment to budget deficit.

\hspace{1cm}27 At first sight, one might even argue that, according to the inequalities (96) - (98), the equilibrium is a stable node for $\alpha > 1$, too. But this would not take into account the fact that the differential equations were derived under the assumption that the aggregate production function is linear homogenous.
Remarkably, the stability conditions are essentially the same as in the original Solow model. Hence, for stability it does not make a difference whether households’ savings are invested in one or several types of capital, fully or partly, as long as the saving rates and corresponding investment rates are exogenous.28

In order to compare the approach of G2 more directly with ours we now derive the differential equation for the debt-to-national-income ratio from the model presented above where the number of new government bonds traded is determined by the demand of private households (see eqs (14) and (10)):

\[ \dot{B} = B^D = S_{pr}^D = s_{pr}^g (1 - \tau)(\dot{Y} + \tau B) \]  

(99).

By inserting eq. (99) into eq. (90) the following differential equation for the debt to national income ratio can be derived:

\[ \dot{b} = s_{pr}^g (1 - \tau) - (1 - s_{pr}^g (1 - \tau)) r_b b - (\dot{Y} - r_B) b \]

\[ = s_{pr}^g (1 - \tau) - (\dot{Y} - s_{pr}^g (1 - \tau) r_B) b \]

(100).

Thus, by using the steady state values of \( \dot{Y} \) and \( r_B \) and assuming \( \dot{Y} (b^*) = (r_B) b^* = 0 \), from eqs (69), (95) and (100) we obtain the following linearized (hence superscript l) differential equation for the debt ratio which is equivalent to the Taylor approximation according to eq. (92) for \( b^* = 0 \) and \( \dot{Y} (b^*) = (r_B) b^* = 0 \):

\[ \dot{b}^l = s_{pr}^g (1 - \tau) - \frac{(\delta + n)s_{pr}^f}{s_{pr}^g + (1 - \sigma)(1 - \alpha)} b^l \]

(101).

Obviously, for \( 0 < \alpha, \sigma, \tau, s_{pr}^g, s_{pr}^f < 1 \), according to eq. (101) the resulting equilibrium debt ratio is stable, as long as \( (\delta + n) \) is positive, the same result as derived above (see inequalities (96) - (98)). From eq. (100) we see that for the equilibrium debt ratio to be stable the growth rate of national income does not have to exceed the rate of interest on government debt, but just the tiny fraction \( s_{pr}^g (1 - \tau) \) of it. In a stable steady state the rate of interest on government debt may exceed the growth rate of national income and hence the government’s solvency constraint may be satisfied.

Thus the answer to the question whether in general a constant debt-to-GDP ratio can only be stable in a growing economy if the growth rate of real GDP and hence real government debt exceeds the real rate of interest on government debt, which in turn leads to a violation of the government’s solvency constraint, is no.

From eqs (22), (65), and (69) we can finally derive the following linearized relation of the primary deficit ratio \( d^l \) and the debt ratio:

\[ d^l = s_{pr}^g (1 - \tau) - \frac{\left(1 - s_{pr}^g (1 - \tau)(\delta + n)(1 - \alpha)(1 - \alpha)\right)}{s_{pr}^f + s_{pr}^g (1 - \alpha)(1 - \alpha)} b^l \]

(102).

According to eq. (102), for a constant rate of interest on government debt at its steady state level \( r_B^* \), an increase in the debt ratio implies a fall in the primary deficit ratio in order to satisfy the government’s liquidity constraint.

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28 How stability is affected if investment functions, that are independent from saving, are introduced in the Solow model is treated in Nikaido 1980. However, this is a study of disequilibrium dynamics in the sense of Hahn and Matthews 1964.
Figure 7 (a) and (b) show how the debt ratio approaches equilibrium for two values of the safety discount factor, starting from an initial value above equilibrium. In Figure 7 (a) the debt ratio stays below its upper threshold $b^*$, in Figure 7 (b) above. Hence, in Figure 7 (a) the government’s solvency constraint is satisfied ($\dot{Y}^* < r_B^*$), but not in Figure 7 (b) ($\dot{Y}^* > r_B^*$). Nonetheless, the steady state is asymptotically stable in both cases, as from $b(0) > b^*$ the debt ratio steadily declines ($\dot{b}(t) \leq 0$) (see Figure 7 (c) and (d)). But Figure 7 (c) and (d) also show that $\dot{b}^l$, unsurprisingly, is a much better approximation of $\dot{b}$ than $\dot{b}^G$, as the latter simply neglects $d_p^G(b^*) = -\left(1-s^G_p(1-\tau)\right)\dot{b}^G$. This neglect does not change the qualitative outcome with respect to stability, as long as $\dot{Y}^* > r_B^*$, but it is very misleading if $\dot{Y}^* < r_B^*$ (see Figure 7 (c)).
In Figure 7 (e) and (f) the differential equation for the debt ratio following from this model ($\dot{b}$) and the corresponding public deficit ratio ($\dot{b}/Y$), their linearized versions ($b^l$ and ($\dot{b}/Y)^l$) as well as their versions according to $G2$ ($b^G$) and (($\dot{b}/Y)^G$) for two values of the safety discount factor are displayed. Again, a first glance suffices to notice that $b^l$ and ($\dot{b}/Y)^l$ are much better approximations than $b^G$ and ($\dot{b}/Y)^G$. Whereas (i) the intercept of the vertical axis of $\dot{b}^l$, which equals the intercept of the linearized relation of the primary deficit ratio and the debt ratio ($d^l$), is independent from the safety discount factor and (ii) the slope of $d^l$ changes so little with a change in the safety discount factor that the corresponding straight lines can rarely be distinguished in Figure 7(e) and (f), both the intercept of the vertical axis of $b^G$, which equals the respective steady state primary deficit ($d^*$), and the slope of $b^G$ change sign.

In any model the best linear approximation is the one that takes into account all the terms of the Taylor approximation of the first order. How much worse the approximation becomes when one or several terms are neglected depends on the size of the neglected terms, here: $d'_b(b^*)(b-b^*)$ and $-b^* \left( \dot{Y}_b(b^*) - (\dot{r}_b)_b(b^*) \right) (b-b^*)$. According to eq. (92) the steady state debt ratio is stable if

$$d'_b(b^*) - (\dot{Y}^* - \dot{r}_b^*) - b^* \left( \dot{Y}_b(b^*) - (\dot{r}_b)_b(b^*) \right) < 0 \quad (103).$$

Hence, as far as stability is concerned, the relative quality of an approximation which neglects more parts of the Taylor approximation depends on the size of those parts of the additionally neglected terms that influence the slope of the differential equation, here: $d'_b(b^*)$, relative to those parts of the terms that are common to both approximations and influence the slope, here: $\dot{Y}^* - \dot{r}_b^*$. The neglect of terms leads to misleading results with respect to stability if this neglect leads to a change in the sign of the slope of the differential equation, as is the case with $G2$ for certain parameter ranges (see Figure 7(e) and (f)). Figure 7(e) and (f) also show that $b^l$ and ($\dot{b}/Y)^l$ approximate the dynamics of the debt ratio and the deficit ratio well, as the green arrows showing these dynamics are almost located on $b^l$ and ($\dot{b}/Y)^l$.

But let us consider the question whether there are circumstances under which $d'_b(b^*) = 0$ holds. This is the case if the demand for government bonds is infinitely elastic and the government chooses to keep the primary deficit ratio constant. In this case, the debt dynamics are determined by the financial needs of government, whereas in the model presented here the public debt dynamics are determined by the demand of private households for government bonds. In the latter case public expenditures on goods and services are adjusted to satisfy the liquidity constraint if the government’s interest payments are changing with public debt (see $d^l$ in Figure 7 (e) and (f)), whereas this is not the case with the former (see $d^G = d^*$ in Figure 7 (e) and (f)). Here it is assumed that the government can keep the ratio of the primary deficit to national income constant even if its interest payments increase. In the former case it is assumed that an increase in public interest payments leads to an increase in the public deficit by the same amount (see ($\dot{b}/Y)^G$ in Figure 7 (e)). In the latter case an increase in public interest payments just leads to an increase in the demand for new government bonds according to the marginal propensity to invest in government bonds after tax $s^G_{pr}(1-\tau)$ (see ($\dot{b}/Y)^l$ in Figure 7 (e)). The lion’s share of an increase in public interest payments $(1-s^G_{pr}(1-\tau))$ has to be compensated by a decrease in the ratio

\[29\text{ See on the importance of demand for government bonds concerning debt sustainability Fisher 2012.}\]
of the primary deficit to national income. Hence, the positive feedback and hence destabilizing effect of an increase in the interest rate on public debt on the dynamics of public debt is much lower in the presented model than in the models by G2.

Public deficits financed exclusively by issuing new government bonds cannot exceed the demand for newly issued government bonds. In principle, budget deficits could be lower than the demand by households. But in the model presented here it is assumed that the government maximizes its budget deficit given households’ demand for newly issued government bonds. As shown above, this does not create problems for sustainability, as long as households’ propensity to invest in government bonds is constant and as long as the safety discount factor is low enough such that the government’s solvency constraint remains satisfied. Instead, in the models of G2 the public deficit is determined by the government’s decisions on spending and taxing, either without adequately considering the demand side in the market for newly issued government bonds or by unrealistically assuming that the demand for government bonds is infinitely elastic.\(^\text{30}\) In this case there would never be a sovereign debt crisis: An ever increasing debt ratio and correspondingly increasing public deficit ratio would not create any problems, as the increasing supply of newly issued government bonds would just be absorbed by investors.

The size of the impact of a change of the debt ratio on the primary deficit ratio is to be determined empirically. Fortunately, there exists a growing body of empirical literature on so-called fiscal reaction functions (see e.g. Greiner et al. 2007, Bohn 2008, Mendoza and Ostry 2008, Ghosh et al. 2013, Weichenrieder and Zimmer 2014, Cevik and Teksoz 2014). Even if the notion ‘fiscal reaction function’ indicates that the authors seem to assume that the budget deficit and hence the primary deficit is mainly determined by the government, not by the demand for newly issued government bonds, the estimates of the size of the impact of a change of the (lagged) debt ratio on the primary deficit ratio can be used nonetheless, as they are just based on correlations, not on causalities. Greiner et al. 2007 generate statistically significant estimates for France, Germany, Italy, Portugal, and the USA for the decades before 2003 in the range of about -0.15, a value that makes sure that inequality (103) is fulfilled. Based on a panel dataset covering the period 1990–2012, and hence the Great Recession as well as the Euro Crisis, for 49 advanced and emerging market economies, Cevik and Teksoz 2014 generate statistically significant estimates of the size of the impact of a change of the lagged debt ratio on the primary deficit ratio which are much smaller in absolute terms, namely around -0.01. Hence, inequality (103) is not fulfilled if the rate of interest on government debt exceeds the GDP growth rate by just more than one percentage point. Thus, the softening of the government’s liquidity constraints in various countries due to purchasing programs of government bonds by Central Banks, which helped to increase the difference between the GDP growth rate and the rate of interest on government bonds, also shows up in these estimates.

Having found more than one definition of sustainability of public debt dynamics in the literature, the question arises what their relationship is. As the solvency constraint refers to the limit of the present value of public debt when time goes to infinity this is certainly a long term concept of sustainability. Instead, stability of the steady state refers to the medium term. Even if the sixty percent threshold of the Maastricht treaty for the public debt ratio was not derived from any model, but was just the average of debt ratios in those countries that were considered probable member states of the European Monetary Union in the beginning of the 1990s, it has gained a certain prominence in public discussions, at least in Europe. Hence, it is taken into account in Table 1 and serves to define subcategories.

\(^{30}\) However, this may be not too unrealistic in an open economy with sufficiently high foreign demand for government bonds. Examples from the recent past might be US as well as German government bonds. Furthermore, the demand for government bonds may be (almost) infinitely elastic if the Central Bank buys them in large amounts, as has been the case in various countries since the outbreak of the Great Recession.
In the presented model the steady state is stable under very general conditions. Hence, the public debt dynamics are medium term sustainable. In the following two figures we see the dynamic equilibrium processes leading to the stable steady state for two parameter constellations which only differ in the values of households’ propensity to invest in government bonds, the safety discount factor, and in the initial values of private and public capital per capita.

The values of households’ propensity to invest in government bonds are selected in such a way that the steady state value of the public debt ratio lies above the Maastricht criterion of 60% in Figure 8, and below in Figure 9, whereas the values of the safety discount factor in such a way that the steady state value of the rate of interest on public debt is higher than the steady state value of the growth rate of public debt in Figure 8, and lower in Figure 9. In Figure 8 the initial value of the public debt ratio is above its steady state value, in Figure 9 below.

Hence, in Figure 8 we consider the case of medium and long term sustainable public debt dynamics with increasing debt ratio, steady state primary surplus and hence satisfied government’s solvency constraint, as well as violated Maastricht debt ratio criterion, in Figure 9, instead, the case of medium term Maastricht sustainable public debt dynamics with decreasing debt ratio, steady state primary deficit and hence violated government’s solvency constraint, as well as satisfied Maastricht debt ratio criterion.

In graph (e) of both figures we see how the primary-deficit-to-national-income ratio and the government-interest-payments-to-national-income ratio change relatively strongly compared to the almost unchanged deficit-to-national-income ratio during the adjustment process. The deficit-to-national-income ratio changes so little during the adjustment process, because the primary-deficit-to-national-income ratio and the government-interest-payments-to-national-income ratio move in opposite directions. As their changes are of almost equal size, their impacts on the deficit-to-national-income ratio almost cancel out.

At first sight one might think that the public debt dynamics look much better in Figure 9 than in Figure 8, as the debt ratio declines in Figure 9 (d), whereas it rises in Figure 8 (d), and the deficit-to-national-income ratio is consistently lower in Figure 9 (e) than in Figure 8 (e). But this view neglects the fact that the upper thresholds for the public deficit ratio and the public debt ratio are considerably lower in Figure 9 than in Figure 8, mainly due to the higher safety discount factor in Figure 9 than in Figure 8. But in Figure 9 (d) the (declining) debt ratio always exceeds its upper threshold, whereas in Figure 8 (d) the (increasing) debt ratio always stays below its upper threshold. Hence, in Figure 8 (e) the initial primary deficit turns into a primary surplus, whereas in Figure 9 (e) the primary deficit lasts forever. Accordingly, after an initial increase, the present value of public debt falls to zero in Figure 8 (h), whereas it rises without any bound in Figure 9 (h).

<table>
<thead>
<tr>
<th>Stable steady state debt ratio &gt; 0</th>
<th>Government’s solvency constraint satisfied</th>
<th>Government’s solvency constraint violated</th>
</tr>
</thead>
<tbody>
<tr>
<td>Medium and long term sustainable public debt dynamics</td>
<td>Medium term sustainable but long term unsustainable public debt dynamics</td>
<td></td>
</tr>
</tbody>
</table>

| 0<Stable steady state debt ratio ≤ 60% | Medium and long term Maastricht sustainable public debt dynamics | Medium term Maastricht sustainable but long term unsustainable public debt dynamics |

| Unstable steady state debt ratio>0 | Long term sustainable public debt dynamics | Unsustainable public debt dynamics |

Table 1: Categories of sustainable and unsustainable public debt dynamics
Figure 8: Medium and long term sustainable public debt dynamics with increasing debt ratio and violated Maastricht debt ratio criterion

\[
\begin{align*}
\kappa(0) &= 1.1, \lambda(0) = 0.3, \gamma = 0.02, \alpha = 0.1, \beta = 0.7, \tau = 0.2, \gamma = 0.02, \sigma = 0.5, \lambda = 1
\end{align*}
\]
If we lowered private households’ propensity to invest in government bonds in Figure 8 to the level that is assumed in Figure 9 the Maastricht criterion would also be fulfilled. This shows that, for a specific set of parameter constellations, the steady state debt ratio can be stable (which we termed medium term sustainable public debt) and, simultaneously, the government’s solvency constraint can be satisfied (which we termed long term sustainable public debt). This set of parameter constellations can be further
divided into two subsets: for one the Maastricht criterion for the public debt ratio is fulfilled, for the other not. Thus, the Maastricht criterion for the public debt ratio cannot be justified by theoretical considerations of the sustainability of public debt.

At least in principle, the upper thresholds for the public deficit ratio and the debt ratio according to eqs (86) and (87) could serve as alternatives to the arbitrary Maastricht criteria. As it might be difficult to observe the determinants of the upper thresholds, a primary budget surplus could be used instead. For this is a necessary and sufficient condition for the long term sustainability of a positive and medium term sustainable steady state public-debt-to-national-income ratio.

In the light of the general elections in Greece in the beginning of 2015 and their political repercussions in other countries of the Euro Area one might add another criterion of the sustainability of public debt, namely one of political economy. Long term sustainability of public debt, hence a primary surplus requires a sufficiently high rate of interest on government bonds, hence a sufficiently low safety discount factor. From Figure 10 we see that due to the liquidity constraint a decrease in the safety discount factor, hence an increase in the steady state rate of interest on government bonds leads to an increase in the primary surplus-to-GDP ratio \((-d')\) and the public interest payments-to-GDP ratio \((I_B^n)\), as well as to a decrease in the government expenditures on goods and services-to-GDP ratio \(((I_g + C_g)/Y)\). Given the outcome of the Greek elections, it seems reasonable to assume that a sufficiently large minority of the population is not willing to vote for a government that uses more than a fraction \(\varepsilon\) of tax revenue to finance interest payments on outstanding debt. Under this assumption a minimum threshold of the safety discount factor \(\sigma_{Min}^\varepsilon\) can be computed that makes sure that not more than the fraction \(\varepsilon\) of tax revenue is used to finance interest payments on outstanding debt:

\[
\sigma_{Min}^\varepsilon = 1 + \frac{s_{pr}^g}{1 - \alpha} \left(1 - \frac{1}{(1 - s_{pr}^g)(1 - \tau) + \varepsilon \tau} \right) \quad (104).
\]

By inserting eq. (104) into eq. (69) we derive the corresponding steady state rate of interest on government bonds:

\[
\left(\frac{r_B^*}{s_{pr}^g} \right)_{Max} = \left(\delta + \alpha \right) \left(1 + \frac{(1 - \varepsilon)\tau}{s_{pr}^g(1 - \tau)} \right) \quad (105).
\]

(a) With regard to the safety discount factor

(b) With regard to the steady state rate of interest on government bonds

Figure 10 shows that there is a ‘sustainability corridor’ of the safety discount factor with \(\sigma_{Min}^\varepsilon\) and \(\sigma_{Max}^\varepsilon\) as boundaries as well as a corresponding ‘sustainability corridor’ of the steady state rate of interest on government bonds.
government bonds with \((r^*_6)_{\text{Min}}\) and \((r^*_6)_{\text{Max}}\) as boundaries. To \((r^*_6)_{\text{Min}}\) corresponds the upper threshold of the economically long term sustainable primary deficit-to-GDP ratio, namely zero, whereas the (negative) lower threshold of the politically sustainable primary deficit-to-GDP ratio \(d_{\text{Min}}\) corresponds to \((r^*_6)_{\text{Max}}\).

Within the ‘sustainability corridor’ both the solvency constraint is satisfied and the government’s primary surplus is (expected to be) politically sustainable. The various actions undertaken to reduce the rate of interest on Greek government bonds since the outbreak of the crisis in 2010 can be viewed as measures to keep the rate of interest on Greek government bonds within the ‘sustainability corridor’. But obviously, at least for the Samaras government, the rate of interest on government bonds turned out not to be politically sustainable.

5. Concluding Remarks

The question that motivated the presented research was whether in general public debt can be sustainable in the sense that the equilibrium debt ratio can be stable and, simultaneously, the government’s solvency constraint can be satisfied for a set of parameter constellations. To this end a simple neoclassical growth model of the Solow type (Solow 1956) was presented that includes government debt. With the help of this model it was shown that public debt can be sustainable for a set of parameter constellations. This set of parameter constellations can be further divided into two subsets: for one the Maastricht criterion for the public debt ratio is fulfilled, for the other not. Thus, the Maastricht criteria for the public deficit and the public debt ratio cannot be justified by theoretical considerations of public debt sustainability. What can be justified instead as a criterion of public debt sustainability is a primary surplus, for which, however, there seems to exist a politically determined upper limit.

A shortcoming of the analysis is that the propensities to save and the safety discount factor are treated as exogenous parameters. These two assumptions lead to the global stability of the steady state. Preliminary results of current research show that the endogenization of the propensities to save and the safety discount factor may destabilize the steady state. More general restrictions of the analysis are that we do not deal with a monetary and open economy including a financial sector that creates money through credit creation. The investigation of whether the introduction of a money creating financial sector changes the results presented above is left for future research.

REFERENCES


