Can Public Debt Be Sustainable?*

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Abstract

The sustainability of government’s financial situation has two aspects: solvency and liquidity. The government is solvent if it satisfies its intertemporal budget constraint. In the steady state of a growing economy, the government is solvent if the public-debt-to-GDP ratio is constant and the real interest rate on government debt exceeds the growth rate of real GDP. The government is liquid if its instantaneous budget constraint is satisfied. According to Gärtner 2009, for a constant primary-deficit-to-GDP ratio, this steady state debt-to-GDP ratio is unstable if the real interest rate on government debt exceeds the growth rate of real GDP. Hence, there seems to be a tension between sustainability concerning government’s solvency and sustainability concerning government’s liquidity if the latter is defined as stability of the steady state debt-to-GDP ratio. This leads to the main research question of this paper: Is it true that in a growing economy a constant debt-to-GDP ratio can only be stable if the government’s solvency constraint is violated? Or put differently: Is it true that public debt cannot be sustainable? Obviously, one counterexample is enough to show that a proposition does not hold in general. Such a counterexample based on the Solow growth model is presented in the paper. Hence, in general, sustainability of public debt is possible in the sense, that both the solvency constraint and the liquidity constraint with a stable steady state debt-to-GDP ratio are satisfied for a set of parameter constellations.

1. Introduction

The most fundamental concept of the sustainability of public debt seems to be that the government remains solvent, i.e. that it satisfies its intertemporal budget constraint, named in the following: solvency constraint. As follows e.g. from Romer 2012 p. 586f, this is the case if the real interest rate on real government debt is always positive and larger than the rate of growth of real government debt. In the steady state of a growing economy, this means that the government’s solvency constraint is satisfied if the public-debt-to-GDP ratio is constant and the real interest rate on government debt exceeds the growth rate of real GDP.

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A government is liquid if its instantaneous budget constraint\(^1\), named in the following liquidity constraint, is satisfied. In case of public deficits, satisfying the government's liquidity constraint requires a sufficient demand for newly issued government bonds if monetization of public debt is excluded. The government's instantaneous budget constraint translates into a differential equation for the debt-to-GDP ratio from which an equilibrium debt-to-GDP ratio can be computed (see e.g. Gärtner 2009 p. 394f and de Grauwe 2014 annex; in the following: G2). According to Gärtner 2009 p. 395f, for a constant primary-deficit-to-GDP ratio, this steady state debt-to-GDP ratio is unstable\(^2\) if the real interest rate on government debt exceeds the growth rate of real GDP.

Hence, there seems to be a tension between sustainability concerning government's solvency and sustainability concerning government's liquidity if the latter is defined as stability of the steady state debt-to-GDP ratio. This leads to the main research question of this paper: Is it true that in a growing economy a constant debt-to-GDP ratio can only be stable if the government's solvency constraint is violated? An affirmative answer would imply that, by its very nature, public debt is unsustainable and, hence, has to be avoided. This would amount to a theoretical justification of a public debt brake.

Obviously, one counterexample is enough to show that a proposition does not hold in general. Such a counterexample is presented here. To this end the simple Solow growth model (Solow 1956) is generalized in order to include public capital and public debt as well as the government's liquidity and solvency constraint. A growth model of the Solow type is chosen because its relevant steady state is asymptotically stable. Hence, the conditions under which the originally stable steady state becomes unstable can be studied. This research strategy obviously could not be followed if the starting point were a model whose steady state is either asymptotically unstable as a growth model of the Harrod-Domar type or a saddle point as an optimal growth model of the Ramsey type.

Presenting a model with a positive steady state debt-to-GDP ratio, where the government is both solvent and liquid in the sense of resilience to shocks and, hence, where its public debt is sustainable, does not mean to deny the possibility of unsustainable public debt. The model presented can thus form the base for finding out deeper reasons for the occurrence of fiscal crises.\(^3\) In the model private households' propensity to invest in government bonds is assumed to be an exogenous parameter just as the safety discount factor. These rather strong assumptions can be justified by three considerations: (i) they are in line with the original Solow growth model, where the saving rate is assumed to be exogenous;\(^4\) (ii) this is a paper on the principal possibility of sustainable public debt, not on public debt crises including runs on government bonds; (iii) in another paper by the author both private households' propensity to invest in government bonds and the safety discount factor will be endogenized.

The remainder of the paper is organized as follows: In section 2 the Solow growth model with public debt is presented. In section 3 we will turn to the question whether it is really true that in a growing economy a constant debt-to-GDP ratio can only be stable if the government's solvency constraint is violated. Furthermore the concept of a political-economic sustainability corridor is introduced. Section 4 concludes.

2. A Simple Solovian Growth Model with Public Debt

In a longer version of this paper (Englmann 2015) a Simple Solovian growth model with public debt, public capital, exogenous propensity to invest in government bonds, and exogenous safety discount

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\(^1\) In the following time is assumed to be continuous.

\(^2\) Now and in the following, by stable we mean asymptotically stable in the sense that after a shock a system returns to its equilibrium (see e.g. Gandolfo 2010 pp. 331ff). Contrary to e.g. European Central Bank 2011 p. 64, by stable we do not mean constant.

\(^3\) First steps in this direction are undertaken in an accompanying paper by the author.

\(^4\) See for a theoretical justification Akerlof's Presidential Address at the Annual Meeting of the American Economic Association 2007 (Akerlof 2007, especially pp 13ff.).
factor is presented. In this section the necessary equations are repeated with just a few explanations. For more details see Englmann 2015.

Following Aschauer 1989 public capital \( K_g \) is taken into account in the aggregate the production function of the Cobb-Douglas type:

\[
Y^S = \gamma A^\alpha L^\beta K_g^{1-\alpha-\beta}, \quad \gamma \geq 1
\]

where \( Y \) denotes aggregate net production of goods and services produced in the economy, \( A \) technical efficiency, \( L \) labor input, \( K_{pr} \) private capital. For the sake of simplicity, the dependence of the variables on time is omitted in general. Total factor productivity grows at the rate of technical progress \( \delta \), labor supply at the natural rate \( n \). Labor supply is inelastic with respect to the real net wage. Capital and labor are fully employed. Aggregate supply of goods and services equals aggregate demand consisting of private \( (C_{pr}) \) and public consumption \( (C_g) \) as well as of net private \( (I_{pr}) \) and net public investment \( (I_g) \).

Private consumption depends on available income of households \( (Y - T + r_B B) \)

\[
C_{pr} = c_{pr} (Y - T + r_B B)
\]

where \( c_{pr} \) denotes households' propensity to consume with respect to disposable income, \( T \) net transfer payments from households to government (taxes plus social contributions from private households minus government’s transfer payments to private households), and \( r_B \) the rate of interest on government bonds \( B \). The non-consumed part of households' disposable income is saved and one part of savings \( S^f_{pr} \) is invested in bonds/equities issued by private firms

\[
S^f_{pr} = s^f_{pr} (Y - T + r_B B)
\]

the other in government bonds

\[
S^g_{pr} = s^g_{pr} (Y - T + r_B B)
\]

In the following, for brevity's sake \( T \) will be called taxes, and the corresponding rate \( \tau \) tax rate.

Taxes depend on the tax rate \( \tau \) and gross household income \( (Y + r_B B) \), which flows from private firms as labor and capital income and from government as interest payments on government bonds

\[
T = \tau (Y + r_B B)
\]

Savings that flow to private firms \( S^f_{pr} \) are used to finance private firms' net investments \( I_{pr} \) in private real capital \( K_{pr} \)

\[
I_{pr} = K_{pr} = S^f_{pr}
\]

Savings that flow to government \( S^g_{pr} \) are used to buy new government bonds \( B^D \) which finance the government’s budget deficit \( B \):

\[
\dot{B} = B^D = S^g_{pr}
\]

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\(^5\) This assumption is in line with von Weizsäcker 2014.

\(^6\) Here and in the following we use the following abbreviations: \( \dot{x} = dx / dt \) and \( \ddot{x} = \ddots \).
According to eq. (7) the government supplies additional government bonds that private households demand depending on their disposable income and risk preferences. Government budget deficits are determined by the portfolio choices of private households just as firms' investments, i.e. firms' deficits. The corresponding public-deficit-to-national-income ratio follows by simple division of eq. (7) by national income:

$$\frac{\dot{B}}{Y} = S_{\text{pr}}^{g}$$  \hspace{1cm} (8).$$

The government's liquidity constraint is:

$$T + \dot{B} = C_{g} + I_{g} + \tau_{g}B$$  \hspace{1cm} (9).$$

Excluding monetization of public debt, the government's total revenue consists of taxes plus net increase in government debt $\dot{B}$. This revenue is used to finance public expenditures on consumption $C_{g}$ and investment $I_{g}$ plus interest payments on outstanding government debt $\tau_{g}B$.

The government's primary deficit $D$ is defined as follows:

$$D = C_{g} + I_{g} - T$$  \hspace{1cm} (10).$$

and the corresponding ratio of the primary deficit to national income $d$ as:

$$d = \frac{D}{Y} = \frac{C_{g} + I_{g} - T}{Y}$$  \hspace{1cm} (11).$$

From eqs (7) and (10) we obtain:

$$\dot{B} = D + \tau_{g}B$$  \hspace{1cm} (12).$$

With the public-debt-to-national-income ratio $b$

$$b = \frac{B}{Y}$$  \hspace{1cm} (13).$$

we get from eqs (11) - (13):

$$d = (\dot{B} - \tau_{g})b$$  \hspace{1cm} (14).$$

For $b > 0$ there is a primary deficit ($d > 0$) if the rate of change of public debt exceeds the real rate of interest on public debt, and there is a primary surplus if the real rate of interest on public debt exceeds the rate of change of public debt.

From eqs (12), (13), (7), (5), and (4) we can also obtain another expression for the primary deficit ratio, namely:

$$d = S_{\text{pr}}^{g}(1-\tau) - \tau_{g} \left(1 - S_{\text{pr}}^{g}(1-\tau)\right)b$$  \hspace{1cm} (15).$$

Hence, the primary debt ratio does not only depend on households' propensity to invest in government bonds and the tax rate, but on the ratio of government's interest payments to national income as well, as the latter influence households' disposable income.

The budget deficit can be used to finance government's net investments in public real capital $K_{g}$ or a part of them as well as to finance a part of public consumption or interest payments on outstanding government debt. We introduce the public-investment-to-budget-deficit ratio $\lambda$, that is supposed to be set by the parliament when it decides on the budget.
In case of \( \lambda = 1 \) the so-called Golden Rule of Public Finance is followed (see e.g. Bassetto and Sargent and 2006)\(^7\). For public capital accumulation we obtain:

\[
K_g = I_g = \lambda B; \quad \lambda > 0
\]  

(17).

From eqs (9) and (17) we can deduce:

\[
T = C_g + I_B B + (1 - 1/\lambda) I_g
\]  

(18).

If the Golden Rule of Public Finance is followed, taxes on national income and interest payments on public debt just serve to finance public consumption and interest payments on public debt. From eqs (5) and (18) we get for the share of public consumption in national income:

\[
\frac{C_g}{Y} = \tau - (1 - \tau) I_B B Y - (1 - 1/\lambda) I_g Y
\]  

(19),

and from eq. (5) for the share of taxes in national income:

\[
\frac{T}{Y} = \tau (1 + I_B B Y)
\]  

(20).

From eqs (2), (4), (6), (13), (17) and (20) the following expression can be derived for the share of public consumption in national income:

\[
\frac{C_g}{Y} = 1 - \left(1 - (1 - \lambda) S_{pr}^B (1 - \tau)(1 + I_B b)\right)
\]  

(21).\(^8\)

The share of public consumption in national income is an endogenous variable in this model which is determined by the government’s liquidity constraint.

As in Solow’s growth model we assume perfect competition in the markets for goods and services, labor and private capital. The price level is assumed to be unity. As all incomes flow to households only private households pay income tax. Hence, the rental price of private capital \(R\) equals the marginal productivity of private capital and the wage rate equals the marginal productivity of labor. Furthermore, we assume that the use of public capital is free of charge. This leads to profits \(\Pi\) even with perfect competition:

\[
\Pi = (1 - \alpha - \beta) Y
\]  

(22).

For the respective rate of profit \(r_{\Pi}\)

\[
r_{\Pi} = \frac{\Pi}{K_{pr}}
\]  

(23)

we can compute:

\[
r_{\Pi} = (1 - \alpha - \beta) \frac{Y}{K_{pr}}
\]  

(24),

and hence, for the overall rate of return on private capital \(r_K\)

\[
r_K = R + r_{\Pi}
\]  

(25).

\(^7\) See in this context also Buiter 2001 and Blanchard and Giavazzi 2004.

\(^8\) It should be noted that eqs (19) and (21) are not independent from each other.
From eqs, (24), and (25) follows:

$$r_K = (1 - \alpha) \frac{Y}{K_{pr}}$$

(26).

Even assuming perfect markets, we allow that the rate of return on private capital may exceed the one on government bonds due to risk and liquidity considerations of private households. This implies that households consider an investment in government bonds less risky and more liquid than an investment in private enterprises. Private households’ risk and liquidity considerations are taken into account by the exogenously given safety discount factor $\sigma$. Hence, we postulate

$$r_B = (1 - \sigma) r_K; \quad 0 \leq \sigma < 1$$

(27).

By assumption, the safety discount factor does not exceed unity. Finally, we define private and public capital-labor ratios in efficiency units:

$$\kappa_{pr} = \frac{K_{pr}}{A L}$$

(28),

$$\kappa_g = \frac{K_g}{A L}$$

(29),

as well as net domestic product per capita in efficiency units:

$$y = \frac{Y}{AL}$$

(30).

The production function can be rewritten by using eqs (30), (28) and (29):

$$y = \frac{Y}{AL} = \gamma\kappa_{pr}^{1-\alpha-\beta}$$

(31).

From eq. (17) we obtain

$$K_g = \lambda B$$

(32)

if $\lambda$ remains unchanged over time, and hence according to eq. (13) for the debt ratio $b$:

$$b = \lambda^{-1} \gamma^{-1} \kappa_{pr}^{\alpha+\beta}$$

(33).

For the interest rate on public debt we can derive from eqs (26) and (27)

$$r_B = \gamma(1-\sigma)(1-\alpha)\kappa_{pr}^{1-\alpha-\beta}$$

(34),

and hence with eqs (34) and (33) for the share of government’s interest payments in national income:

$$r_B b = \lambda^{-1} (1 - \sigma)(1 - \alpha) \frac{K_g}{\kappa_{pr}}$$

(35),

and hence with eq. (20) for the ratio of households’ disposable income to national income:

$$\frac{Y - T + r_B B}{Y} = 1 - \tau + (1 - \tau) r_B b = (1 - \tau) \left(1 + \lambda^{-1} (1 - \sigma)(1 - \alpha) \frac{K_g}{\kappa_{pr}}\right)$$

(36).

From eqs (36), (3), and (4) we can deduce the shares of households’ investment in private and public bonds in national income:

$$\frac{S_{pr}^f}{Y} = S_{pr}^f (1 - \tau) \left(1 + \lambda^{-1} (1 - \sigma)(1 - \alpha) \frac{K_g}{\kappa_{pr}}\right)$$

(37).
From eqs (26) and (31) we derive for the rate of return on private capital:

$$r_K = \gamma(1-\alpha)\kappa_{pr}^{\beta-1} \kappa_g^{1-\alpha-\beta}$$

(39).

Finally the following system of differential equations is obtained:

$$\dot{\kappa}_{pr} = s_{pr}^f (1-\tau)\gamma \left( \kappa_{pr}^{\beta-1-\alpha-\beta} + \lambda^{-1}(1-\sigma)(1-\alpha)\kappa_{pr}^{-1} \kappa_g^{-2-\alpha-\beta} \right) - (\delta + n)\kappa_{pr}$$

(40),

and:

$$\dot{\kappa}_g = \lambda s_{pr}^g (1-\tau)\gamma \left( \kappa_{pr}^{\beta-1-\alpha-\beta} + \lambda^{-1}(1-\sigma)(1-\alpha)\kappa_{pr}^{-1} \kappa_g^{-2-\alpha-\beta} \right) - (\delta + n)\kappa_g$$

(41).

The steady state ( \( \kappa_{pr}^*, \kappa_g^* \)) is derived by setting both eqs (40) and (41) equal to zero. By dividing the two resulting equations, we can see that the following relationship holds in the steady state:

$$\kappa_g^* = \lambda \frac{s_{pr}^g}{s_{pr}^f} \kappa_{pr}^*$$

(42).

We define the ratio of the propensities to save:

$$s_{pr}^{fg} = \frac{s_{pr}^f}{s_{pr}^g}$$

(43).

By inserting eq. (42) into eqs (37) and (38) we can derive the steady state values of the shares of households' investment in private and public bonds in national income:

$$\bar{s}_{pr}^f = s_{pr}^f (1-\tau)\gamma \left( 1 + (1-\sigma)(1-\alpha)\left( s_{pr}^{fg} \right)^{-1} \right)$$

(44)

and

$$\bar{s}_{pr}^g = s_{pr}^g (1-\tau)\gamma \left( 1 + (1-\sigma)(1-\alpha)\left( s_{pr}^{fg} \right)^{-1} \right)$$

(45).

According to eq. (8) the households’ propensity to save and invest in government bonds with respect to national income \( \bar{s}_{pr}^g \), equals the steady state public-deficit-to-national-income ratio:

$$\left( \frac{B}{Y} \right)^* = \bar{s}_{pr}^g = s_{pr}^g (1-\tau)\gamma \left( 1 + (1-\sigma)(1-\alpha)\left( s_{pr}^{fg} \right)^{-1} \right)$$

(46).

From eq. (46) we see that the steady state public deficit ratio is constant.

We obtain the steady state values of private and public capital per capita by inserting eq. (42) into eq. (40):\(^9\)

\(^9\) As in the original Solow model there is a second steady state, namely \( \kappa_g = \kappa_{pr} = 0 \). This steady state is an unstable node.
and finally, again using eq. (42):

$$\kappa^*_{pr} = \left[ \gamma \lambda^{1-\beta} \left( \frac{f_g}{s_{pr}} \right)^{\beta} \frac{s_g}{s_{pr}} \right]^{\gamma/\alpha}$$

Figure 1 shows how the steady state values of the private and public capital-labor ratios can be determined graphically in the three-dimensional Solow Diagram in the intersection point of the two differential equations and the $\kappa_{pr} = \kappa_g = 0$-plain.

By inserting eqs (47) and (48) into eq. (31) and taking account of eq. (45) the steady state value of national income per capita in efficiency units follows:

$$y^* = \left[ \gamma \lambda^{(1-\alpha-\beta)} \left( \frac{f_g}{s_{pr}} \right)^{\beta} \left( \frac{s_g}{s_{pr}} \right)^{(1-\alpha)} \right]^{\gamma/\alpha}$$

In analogy to the simple Solow model (Solow 1956), according to eq. (49), the steady state national income per capita increases with a proportional increase in the propensities to save and a decrease in the equilibrium growth rate $(\delta + n)$. Furthermore, the steady state national income per capita in efficiency units increases with increases in the public-investment-to-budget-deficit ratio $\lambda$ and with decreases in the tax rate $(\tau)$ and the safety discount factor $(\sigma)$.

Obviously, in the steady state the growth rates of national income, private and public capital as well as public debt coincide with the growth rate of labor supply in efficiency units:

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**The parameter values in the various figures are based on rough estimates for Germany.**
\[ \dot{Y}^* = \dot{K}_g^* = \dot{K}_p^* = \dot{B}^* = \delta + n \]  

(50).

By inserting eqs (47) and (48) into eq. (33), and using eq. (45), we get for the steady state value of the ratio of public debt to national income:

\[ b^* = \frac{s_{pr}^g (1 - \tau)}{\delta + n} \left( 1 + (1 - \sigma)(1 - \alpha) \left( s_{pr}^g \right)^{-1} \right) = \frac{s_{pr}^g \delta + n}{\delta + n} \]  

(51).

Eq. (51) shows the well-known result that the steady state value of the debt ratio is equal to the steady state public deficit ratio divided by the steady state growth rate of national income. Under the additional assumptions \( s_{pr}^g > 0 \) and \( \delta + n > 0 \) the steady state debt ratio \( b^* \) is also positive.

By inserting eq. (42) into eq. (35) we derive for the steady state value of the ratio of public interest payments to national income:

\[ f_{B}^* = \frac{s_{pr}^g (1 - \sigma)(1 - \alpha)}{s_{pr}^f} \]  

(52).

For the steady state value of the ratio of public interest payments to national income to be smaller than unity it is sufficient that households invest at least as much in private firms as in government bonds.

If we divide eq. (52) by eq. (51) we finally obtain the steady state value of the interest rate on public debt:

\[ f_{B}^* = \frac{(\delta + n)(1 - \sigma)(1 - \alpha)}{s_{pr}^g s_{pr}^f} \]  

(53).

Neither the steady state value of the ratio of public interest payments to national income nor the steady state rate of interest on government bonds depend on the ratio of public investment to budget deficit \( \lambda \).

By using eqs (11), (17), (20), (21), (51) - (53) the equilibrium value of the ratio of the primary deficit to national income \( d \) can be computed as:

\[ d^* = \left[ s_{pr}^f (1 - \tau) - (1 - \sigma)(1 - \alpha) \left( 1 - s_{pr}^g (1 - \tau) \right) \left( s_{pr}^g \right)^{-1} \right] \]  

(54).

It can be positive or negative or zero.

From eqs (14) and (50) we directly get the following relationship between the steady state values of the primary deficit ratio, the debt ratio, and the difference between the growth rate of real national income and the real rate of interest on government bonds:

\[ d^* = (\dot{Y}^* - f_{B}^*) b^* \]  

(55).

For \( b^* > 0 \) we can derive:

\[ d^* \geq 0 \iff \dot{Y}^* \geq f_{B}^* \]  

(56).

Hence, for \( b^* > 0 \), in the steady state there is a primary deficit if the growth rate of real national income exceeds the real rate of interest on public debt, and there is a primary surplus if the growth rate of real national income is smaller than the real rate of interest on public debt.
3. Sustainability of Public Debt Dynamics

We now turn to the question posed above, namely: Is it true that in a growing economy a constant debt-to-GDP ratio can only be stable if the real rate of interest on government debt is lower than the growth rate of real GDP which in turn leads to a violation of the government’s solvency constraint?

Following Romer 2012, p. 586 we write the government’s solvency constraint in the following form:

\[
\lim_{t \to \infty} PVB(t) = \lim_{t \to \infty} B(t)e^{-R_B(t)} \leq 0 \tag{57}
\]

with

\[
R_B(t) = \int_{\nu=0}^{t} r(\nu)d\nu \tag{58},
\]

where \( PVB \) denotes the present value of public debt in \( t = 0 \) and \( R_B(t) \) the compound interest on government bonds from \( t = 0 \) until \( t \). As public debt is positive as long as households’ propensity to invest in government bonds is positive, the present value of public debt has to converge to zero in order to satisfy the government’s solvency constraint in our model. Hence, the rate of change of \( PVB \) has to be negative when time goes to infinity:

\[
\lim_{t \to \infty} (PVB(t)) = \lim_{t \to \infty} \dot{B}(t) - \lim_{t \to \infty} r_B(t) < 0 \tag{59}.
\]

Obviously, if the model economy is not in the steady state right from the start, the steady state values are only approached if the steady state is asymptotically stable. Only under this condition holds:

\[
\lim_{t \to \infty} (PVB(t)) = \dot{B}^* - r_B^* = (PVB)^* < 0 \tag{60}.
\]

Hence, the stability of the steady state also matters for the answer to the question of whether the government’s solvency constraint is satisfied or not. In the steady state the rate of change of public debt equals the steady state growth rate of national income \( (\delta + n) \) according to eq. (50). Thus from (60) follows the minimum threshold of the steady state rate of interest on government bonds \( r_{BMin}^* \) that has to be exceeded in order to satisfy the solvency constraint:

\[
r_{BMin}^* = \delta + n \tag{61}.
\]

Taking into account eq. (53) condition (60) can be rewritten as:

\[
(PVB)^* = (\delta + n) \left(1 - \frac{s_{pr}^B(1-\sigma)(1-\alpha)}{s_{pr}^f s_{pr}^B}\right) < 0 \tag{62}.
\]

According to eq. (62), for \((\delta + n) > 0\) the sign of the steady state rate of change of the present value of public debt depends on all the other parameters that influence the steady state rate of interest on government bonds alone (see the fraction in eq. (62) and taking into account eq. (53)). Hence, in order to fulfil the solvency constraint it is sufficient for the government to realize a primary surplus in the steady state, as long as the steady state is stable.

For \((\delta + n) > 0\) inequality (62) is satisfied if

\[\text{This formulation is more general than the one used by Blanchard et al 1990, p. 12, who assume a constant rate of growth of real GDP and a constant real rate of interest.}\]
\[ g_{pr} \leq \frac{s_{pr}^g (1-\sigma)(1-\alpha)}{s_{pr}^f (\delta + n)} = \rho_B b^* = \frac{B}{Y} \]  

(63),

i.e. if the steady state public-deficit-to-national-income ratio is smaller than the steady state government-interest-payments-to-national-income ratio. According to inequality (63) the steady state government-interest-payments-to-national-income ratio is the upper threshold of the steady state public-deficit-to-national-income ratio \((B/Y)\) that is not to be reached if the government’s solvency constraint is to be satisfied.

Figure 2: The steady state public deficit ratio, its upper threshold, and the Maastricht criterion

\((s_{pr}^g = 0.04; s_{pr}^f = 0.1; \gamma = 1; \alpha = 0.7; \beta = 0.2; n = 0; \delta = 0.02; \lambda = 1)\)

As Figure 2 shows, by setting a sufficiently high tax rate, the government has an instrument to make sure that its solvency constraint is satisfied, even if the safety discount is large.\(^{12}\) For the corresponding upper threshold of the steady state debt ratio we get from inequality (63): 

\[ b^* = \frac{s_{pr}^g (1-\sigma)(1-\alpha)}{s_{pr}^f (\delta + n)} = \rho_B b^* = \frac{B}{Y} \]

(64).

From (56), (60), (63), and (64) follows 

\[ \rho_B \geq \gamma^* \Leftrightarrow \rho_B b^* = \left( \frac{B}{Y} \right)^* \Leftrightarrow 0 \leq d^* \Leftrightarrow 0 \leq (PVB) \Leftrightarrow b^* \leq b^* \]

(65)

From eqs (59) and (62) we observe that satisfying the government’s solvency constraint as a condition for the sustainability of public debt requires the steady state rate of interest on government bonds to exceed the steady state growth rate of national income. But, according to Gärtner 2009 p. 395f and de Grauwe 2014 annex (G2), this condition leads to instability of the steady state debt ratio and hence non-sustainability of public debt. Thus let us have a closer look at their approach.\(^{13}\)

By differentiating the debt ratio with respect to time (see eq. (33)), inserting eq. (9), and taking into account the definition of the primary budget deficit the following differential equation for the debt-to-national-income ratio can be derived:

\[ \dot{b} = \frac{C_g + I_g - T}{Y} - (\dot{Y} - \dot{g})b = d - (\dot{Y} - \dot{g})b \]

(66),

\(^{12}\) Assuming the labor supply to depend on the tax rate might change this conclusion.

\(^{13}\) See in this context Greenlaw et al. 2013, too.
from which – by setting $b = 0^{14} –$ the equilibrium value of $b$ can be directly computed as eq. (55). In principle, $d, \hat{Y},$ and $r_B$ could be functions of $b$. Hence, by linearizing eq. (66) around the equilibrium by means of a Taylor series of the first order we get:

$$b^T = d(b^*) - (\hat{Y}^*(b^*) - r_B(b^*))b + d_b(b^*)(b-b^*) - b^* \left(\hat{Y}_b(b^*) - (r_B)'(b^*)\right)(b-b^*)$$

(67),

where $T$ refers to Taylor and $d(b^*) = d^*, \hat{Y}(b^*) = \hat{Y}^*, r_B(b^*) = r_B^*.$ G2 treat $d, \hat{Y},$ and $r_B$ as exogenous parameters. Hence they (implicitly) assume:

$$d_b(b^*) = \hat{Y}_b(b^*) = (r_B)'(b^*) = 0$$

(68).

Accordingly, G2’s (linear) differential equation for the debt ratio can be written as follows:

$$b^G = d^* - (\hat{Y}^* - r_B^*)b^G$$

(69),

where $G$ refers to G2. As the growth rate and the rate of interest on government bonds do not directly depend on the debt ratio, we maintain the simplifying assumption $\hat{Y}_b^*(b^*) = (r_B)'(b^*) = 0$ as a first approximation, but not $d_b(b^*) = 0$, as the primary deficit ratio directly depends on the debt ratio according to eq. (15):

$$d_b'(b^*) = -(1 - s^G/(1 - \tau))r_B^*$$

(70).

If $d^* < 0$ and hence $\hat{Y}^* < r_B^*$ the resulting differential equation $b^G$ is positively sloped and, thus, the equilibrium seems to be unstable even if it is stable, as will be shown in the following.

By linearizing the differential equations (40) and (41) in the relevant steady state we can compute the discriminant $\Delta$:

$$\Delta = (\hat{o} + n)^2(1 - \alpha)^2 > 0 \ \forall \ \ (\hat{o} + n) \neq 0; \ \alpha \neq 1$$

(71).

Hence, we have two real roots. For the trace $tr$ of the Jacobian matrix we get:

$$tr = -(\hat{o} + n)(1 + \alpha) < 0 \ \forall \ \ (\hat{o} + n) > 0; \ \alpha > 0$$

(72),

and for its determinant $det$:

$$det = (\hat{o} + n)^2 \alpha > 0 \ \forall \ \ (\hat{o} + n) \neq 0; \ \alpha > 0$$

(73).

Because of eqs (71) - (73) the relevant steady state is a stable node (see Gandolfo 2010, p.358), as long as $(\hat{o} + n) > 0$ and $1 > \alpha > 0^{15}.$ Remarkably, the stability conditions are essentially the same as in the original Solow model. Hence, for stability of the steady state it does not make a difference whether households’ savings are invested in one or several types of capital, fully or partly, as long as the saving rates and corresponding investment rates are exogenous.

In order to compare the approach of G2 more directly with ours we now derive the differential equation for the debt-to-national-income ratio from eqs (7), (4)) and eq. (66) as:

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14 de Grauwe calls this the necessary condition for maintaining solvency (de Grauwe 2014). Hence, his use of the term ‘solvency’ is different from mine.

15 At first sight, one might even argue that, according to the inequalities (71) - (73), the equilibrium is a stable node for $\alpha > 1$, too. But this would not take into account the fact that the differential equations were derived under the assumption that the aggregate production function is linear homogenous.
\[ \dot{b} = s^g_{pr} (1 - \tau) - \left( \dot{Y} - s^g_{pr} (1 - \tau)r_B \right) b \]  

(74).

Thus, by using the steady state values of \( \dot{Y} \) and \( r_B \), and by assuming \( \dot{Y}_0(b^*) = (r_B)_0(b^*) = 0 \), from eqs (53), (70) and (74) we obtain the following linearized (hence superscript l) differential equation for the debt ratio which is equivalent to the Taylor approximation according to eq. (67) for \( \dot{d}_b(b^*) \neq 0 \) and \( \dot{y}_0(b^*) = (r_B)_0(b^*) = 0 : \)

\[ \dot{b}^l = s^g_{pr} (1 - \tau) - \frac{(\delta + n)s^f_{pr}}{s^f_{pr} + (1 - \sigma)(1 - \alpha)} b^l \]  

(75).

Obviously, for \( 0 < \alpha, \sigma, \tau, s^g_{pr}, s^f_{pr} < 1 \), according to eq. (75) the resulting equilibrium debt ratio is stable, as long as \( (\delta + n) \) is positive, the same result as derived above (see inequalities (71) - (73)). From eq. (74) we see that for the equilibrium debt ratio to be stable the growth rate of national income does not have to exceed the rate of interest on government debt, but just the tiny fraction \( s^g_{pr} (1 - \tau) \) of it.

Thus the answer to the question whether in general a constant debt-to-GDP ratio can only be stable in a growing economy if the growth rate of real GDP and hence real government debt exceeds the real rate of interest on government debt, which in turn leads to a violation of the government’s solvency constraint, is \textbf{no}. Hence, public debt can be sustainable in the sense that both the solvency constraint and the liquidity constraint are satisfied for a stable steady state debt ratio.

From eqs (15), (50), and (53) we can finally derive the following linearized relation of the primary deficit ratio \( (d^l) \) and the debt ratio:

\[ d^l = s^g_{pr} (1 - \tau) - \frac{1 - s^g_{pr} (1 - \tau)(\delta + n)(1 - \alpha)(1 - \alpha)}{(1 - \tau)(s^f_{pr} + s^g_{pr} (1 - \alpha)(1 - \alpha))} b^l \]  

(76).

According to eq. (76), for a constant rate of interest on government debt at its steady state level \( (r_B^*) \), an increase in the debt ratio implies a fall in the primary deficit ratio in order to satisfy the government’s liquidity constraint.

Figure 3 (a) and (b) show how the debt ratio approaches equilibrium for two values of the safety discount factor, starting from an initial value above equilibrium. In Figure 3 (a) the debt ratio stays below its upper threshold \( b^* \), in Figure 3 (b) above. Hence, in Figure 3 (a) the government’s solvency constraint is satisfied \( (\dot{Y}^* < r_B^*) \), but not in Figure 3 (b) \( (\dot{Y}^* > r_B^*) \). Nonetheless, the steady state is asymptotically stable in both cases, as from \( b(0) > b^* \) the debt ratio steadily declines \( (\dot{b}(t) \leq 0) \) (see Figure 3 (c) and (d)). But Figure 3 (c) and (d) also show that \( \dot{b}^l \), unsurprisingly, is a much better approximation of \( \dot{b}^G \) as the latter simply neglects \( d^l_0(b^*) = - \left( 1 - s^g_{pr} (1 - \tau) \right) r_B^* \). This neglect does not change the qualitative outcome with respect to stability, as long as \( \dot{Y}^* > r_B^* \), but it is very misleading if \( \dot{Y}^* < r_B^* \) (see Figure 3 (c)).
In Figure 3 (e) and (f) the differential equation for the debt ratio following from this model ($\dot{b}$) and the corresponding public deficit ratio ($\dot{B}/Y$), their linearized versions ($\dot{b}^l$) and (($\dot{B}/Y)^l$) as well as their versions according to G2 ($\dot{b}^G$) and (($\dot{B}/Y)^G$) are displayed for two values of the safety discount factor. Again, a first glance suffices to notice that $\dot{b}^l$ and ($\dot{B}/Y)^l$ are much better approximations than $\dot{b}^G$ and ($\dot{B}/Y)^G$, as the green arrows showing these dynamics are almost located on $\dot{b}^l$ and ($\dot{B}/Y)^l$.

In any model the best linear approximation is the one that takes into account all the terms of the Taylor approximation of the first order. How much worse the approximation becomes when one or several terms are neglected depends on the size of the neglected terms. According to eq. (67) the steady state debt ratio is stable if

$$d_{b}(b^*) - (\ddot{Y}^* - r_B^*) - b^* \left( \dot{b}_b^*(b^*) - (r_B^*) \dot{b}_b^*(b^*) \right) < 0$$

(77).
The neglect of terms leads to misleading results with respect to stability if this neglect leads to a change in the sign of the slope of the differential equation, as is the case with $G_2$ for specific parameter ranges (see Figure 3(e) and (f)).

The size of the impact of a change of the debt ratio on the primary deficit ratio is to be determined empirically. Fortunately, there exists a growing body of empirical literature on so-called fiscal reaction functions (see e.g. Greiner et al. 2007, Bohn 2008, Mendoza and Ostry 2008, Ghosh et al. 2013, Weichennieder and Zimmer 2014, Cevik and Teksoz 2014). Greiner et al. 2007 generate statistically significant estimates for France, Germany, Italy, Portugal, and the USA for the decades before 2003 in the range of about -0.15, a value that makes sure that inequality (77) is fulfilled. Based on a panel dataset covering the period 1990–2012, and hence the Great Recession as well as the Euro Crisis, for 49 advanced and emerging market economies, Cevik and Teksoz 2014 generate statistically significant estimates of the size of the impact of a change of the lagged debt ratio on the primary deficit ratio which are much smaller in absolute terms, namely around -0.01. Hence, inequality (77) is not fulfilled if the rate of interest on government debt exceeds the GDP growth rate by just more than one percentage point. Thus, the softening of the government’s liquidity constraints in various countries due to purchasing programs of government bonds by Central Banks, which helped to increase the difference between the GDP growth rate and the rate of interest on government bonds, also shows up in these estimates.

Figure 4 shows the dynamic equilibrium processes leading to the stable steady state for two parameter constellations which only differ in the values of households’ propensity to invest in government bonds, the safety discount factor, and in the initial values of private and public capital per capita. On the LHS we see sustainable public debt dynamics with increasing debt ratio and violated Maastricht debt ratio criterion, on the RHS unsustainable public debt dynamics with decreasing debt ratio and satisfied Maastricht debt ratio criterion. Obviously, the Maastricht criterion does not help to distinguish sustainable from unsustainable public debt dynamics.

At least in principle, the upper thresholds for the public deficit ratio and the debt ratio according to eqs (63) and (64) could serve as alternatives as well as a primary budget surplus. For this is a necessary and sufficient condition for the sustainability of public debt in the sense that both the solvency and liquidity constraint are fulfilled for stable steady state debt and deficit ratios.
Figure 4: Sustainable public debt dynamics with increasing debt ratio and violated Maastricht debt ratio criterion and unsustainable public debt dynamics with decreasing debt ratio and satisfied Maastricht debt ratio criterion

In the light of the general elections in Greece in the beginning of 2015 one might add another criterion of the sustainability of public debt, namely one of political economy. Sustainability of public debt, hence a primary surplus requires a sufficiently high rate of interest on government bonds. From Figure 5 we see that because of the liquidity constraint an increase in the steady state rate of interest on government bonds due to a decrease in the safety discount factor leads to an increase in the primary surplus-to-GDP ratio \((-d^-)\) and the public interest payments-to-GDP ratio \((r_b^b)\), as well as to a decrease in the government expenditures on goods and services-to-GDP ratio \((I_{Cg}/Y)\). Given the outcome of the Greek elections, it seems reasonable to assume that a sufficiently large minority of the population is not willing to vote for a government that uses more than a fraction \(\varepsilon\) of tax revenue to finance interest payments on outstanding debt. Under this assumption a maximum threshold of the interest rate on govern-
ment bonds \( \left( r_b^* \right)^{c}_{\text{Max}} \) can be computed that makes sure that not more than the fraction \( \varepsilon \) of tax revenue is used to finance interest payments on outstanding debt:

\[
\left( r_b^* \right)^{c}_{\text{Max}} = \left( \delta + n \right) \left( 1 + \frac{(1-\varepsilon)\tau}{s^r_{\text{pr}}(1-\tau)} \right)
\]  

(78).

Figure 5 shows that there is a ‘sustainability corridor’ of the steady state rate of interest on government bonds with \( \left( r_b^* \right)^{\text{PV}}_{\text{Min}} \) and \( \left( r_b^* \right)^{c}_{\text{Max}} \) as boundaries. To \( \left( r_b^* \right)^{\text{PV}}_{\text{Min}} \) corresponds the upper threshold of the economically long term sustainable primary deficit-to-GDP ratio, namely zero, whereas the (negative) lower threshold of the politically sustainable primary deficit-to-GDP ratio \( d^c_{\text{Min}} \) corresponds to \( \left( r_b^* \right)^{c}_{\text{Max}} \).

Within the ‘sustainability corridor’ both the solvency constraint is satisfied and the government’s primary surplus is (expected to be) politically sustainable. The various actions undertaken to reduce the rate of interest on Greek government bonds since the outbreak of the crisis in 2010 can be viewed as measures to keep the rate of interest on Greek government bonds within the ‘sustainability corridor’. But obviously, at least for the Samaras government, the rate of interest on government bonds turned out not to be politically sustainable.

### 4. Concluding Remarks

The question that motivated the presented research was whether in general public debt can be sustainable in the sense that the equilibrium debt ratio can be stable and, simultaneously, the government’s solvency constraint can be satisfied for a set of parameter constellations. To this end a simple neoclassical growth model of the Solow type (Solow 1956) was presented that includes government debt. With the help of this model it was shown that public debt can be sustainable for a set of parameter constellations. This set of parameter constellations can be further divided into two subsets: for one the Maastricht criterion for the public debt ratio is fulfilled, for the other not. Thus, the Maastricht criteria for the public deficit and the public debt ratio cannot be justified by theoretical considerations of public debt sustainability. What can be justified instead as a criterion of public debt sustainability is a primary surplus, for which, however, there seems to exist a politically determined upper limit.
REFERENCES


